On the André-Pink-Zannier conjecture.

(joint work with Rodolphe Richard)

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The Manin-Mumford ‘conjecture’

Let $A = \mathbb{C}^n/\Gamma$ be an abelian variety and Σ a set of torsion points. Components of Σ^{Zar} are translates of abelian subvarieties by torsion points i.e of the form $P + B$ where P is a torsion point and B an abelian subvariety.

Such translates are called ‘special subvarieties’.

A natural analogue in the Hermitian (or Shimura case) is the André-Pink-Zannier conjecture.

Weakly special subvarieties.

Abelian case : translates of abelian subvarieties : $Z = B + P$ where B is an abelian subvariety and P a point.

Shimura case : Let S be a Shimura variety.
A subvariety Z is called weakly special if there exists a Shimura subvariety $S' = S_1 \times S_2 \subset S$ such that $Z = S_1 \times \{x\}$ where x is a point of S_2.
(we allow S_2 to be 'empty' in which case Z is called 'special', or S_1 to be empty' in which case Z is a point).

Note the analogy : in the abelian case, A is isogeneous to $B \times B'$ and under this isogeny Z becomes $B \times \{P\}$.

Bi-algebraic point of view.

A very useful point of view (especially from the perspective of Pila-Zannier o-minimal strategy)

Hecke orbits.

A Shimura variety is defined by a Shimura datum (G, X) (here G is a reductive group and X is a certain hermitian symetric domain, homogeneous space under $G(\mathbb{R})$).
One also needs a compact open subgroup $K \subset G(\mathbb{A}_f)$.

$$Sh_K(G, X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$$

May assume $K = \prod_p K_p$.
A point $s \in Sh_K(G, X)$ can be written as $\overline{(h, 1)}$.
The Hecke orbit of s is the set

$$H(s) = \{\overline{(h, g)} : g \in G(\mathbb{A}_f)\}$$

In the case of $\mathcal{A}_g$ ($G=$ symplectic group), a Hecke orbit of s is simply the isogeny class of the abelian variety corresponding to s.

P-Hecke orbits.

Let P be a fixed set of primes, define

$$\mathbb{Q}_P = \prod_{p \in P} \mathbb{Q}_p$$

and

$$G_P = \prod_{p \in P} G(\mathbb{Q}_p)$$

The P-Hecke orbit of s is

$$H_P(s) = \{\overline{(h, g)} : g \in G_P\}$$

This, in the case of $\mathcal{A}_g$ corresponds to the set of abelian varities isogeneous by an isogeny of degree only divisible by primes in P.

The André-Pink-Zannier conjecture.

Let S be a Shimura variety and s a point.
Let Σ be a subset of $H(s)$.

Components of Σ^{Zar} are special.

The conjecture remains open in general, but there are quite general results by Martin Orr.

The P-André-Pink-Zannier conjecture states the same for a subset of $H_P(s)$.

Orr's theorem.

Let $S = \mathcal{A}_g$, s a point and Z a component of the Zariski closure of a subset of $H(s)$. There exists a Shimura subvariety $S' \subset S$ and a decomposition $S' = S_1 \times S_2$ and a subvariety V' of such that $Z = S_1 \times V'$.

Consequence : The André-Pink-Zannier conjecture holds for curves in $\mathcal{A}_g$.
Orr also proved P-André-Pink-Zannier conjecture.

Ingredients of the proof : adaptation of the Pila-Zannier strategy ; o-minimality, Pila-Wilkie, Masser-Wustholtz, hyperbolic Ax-Lindemann and its consequences.

Galois representations.

Let E be a field of finite type, $s = \overline{(h, 1)} \in Sh_K(G, X)(E)$.

Points in $H(s)$ are defined over $\overline{E}$.
Let P be a finite set of primes.

There exists a Galois representation

$$\rho_{h,P} \colon \mathrm{Gal}(\overline{E}/E) \longrightarrow M(\mathbb{A}_f) \cap K \cap G_P$$

Let

$$U_P := \rho_{h,P}(\mathrm{Gal}(\overline{E}/E)) \subset M(\mathbb{A}_f) \cap G_P$$

This is a P-adic Lie subgroup of $M(\mathbb{A}_f) \cap G_P$.
Also let

$$H_P = U_P^{Zar}$$

We say that $\rho_{h,P}$ is of :

1. P-Mumford-Tate type if U_P is open in $M(\mathbb{A}_f) \cap G_P$.
2. P-Tate type if M and H_P^0 have the same centraliser in G_P.
3. satisfies P-semisimplicity if H_P is a reductive group
4. satisfies P-algebraicity if U_P is open in H_P

Remarks :

1. P-M.T type implies P-Tate
2. P-Tate holds for all Shimura varieties of abelian type (Faltings)
3. P-M.T holds for special points
4. P-Tate implies algebraicity

Real weakly special subvarieties.

A subgroup $L \subset G$ is of P-**Ratner class** if its Levi subgroups are semisimple and for every $\mathbb{Q}$-quasi factor F of a Levi, $F(\mathbb{R} \times \mathbb{Q}_P)$ is not compact.
Given $s = \overline{(h, 1)}$, define

$$Z_{L,s} = \{\overline{(l \cdot h, 1)}, l \in L(\mathbb{R})^+\}.$$

We cal a subset $Z = Z_{L,s}$ for some L qnd s a real weakly special submanifold.
Note

$$Z_{L,s} = \Gamma \backslash L(\mathbb{R})^+ / L(\mathbb{R})^+ \cap K_h$$

where K_h is the stabiliser of h in $L(\mathbb{R})^+$.
There is a canonical probability measure on S with support in Z.
The Zariski closure of such a Z is weakly special.

Topological and Zariski P-André-Pink-Zannier

Let s be a point of $S(E)$. For a subset $\Sigma \in H_P(s)$, consider

$$\Sigma_E = \mathrm{Gal}(\overline{E}/E) \cdot \Sigma = \{\sigma(x) : \mathrm{Gal}(\overline{E}/E), x \in \Sigma\}$$

Then

1. if s is of P-Tate type, the topological closure of Σ_E is is a finite union of weakly P-special real submanifolds.
2. if s is of P-Mumford-Tate type, then the topological closure of Σ_E is is a finite union of weakly special subvarieties.
3. if s is of P-Tate type, then the Zariski closure of Σ is a finite union of weakly special subvarities.

The equidistribution theorem.

Let (s_n) be a sequence of points $H_P(s)$ and let

$$\mu_n = \frac{1}{\mid Gal(\overline{E}/E) \mid} \sum_{z \in Gal(\overline{E}/E) \cdot s_n} \delta_z$$

There exists a finite set $Z_1, \ldots, Z_r$ of weakly P-special subvarieties such that μ_n converges to

$$\mu_\infty = \frac{1}{r} \sum_1^r \mu_{Z_i}$$

and for all n large enough,

$$Supp(\mu_n) \subset Supp(\mu_\infty) = \bigcup_1^r Z_i$$

Furthermore, if s is of P-Mumford-Tate type, then each Z_i is a weakly special subvariety.

Very rough sketch of proofs.

Let $s = \overline{(s, 1)}$ and U_P as before the image of Galois. It's a compact group.
Write $s_n = \overline{(h, g_n)}$ with $g_n \in G_P$.
We 'lift the situation' to $G(\mathbb{R} \times \mathbb{Q}_P)$.
Let

$$\Gamma = (G(\mathbb{R}) \cdot G_P \cdot K) \cdot G(\mathbb{Q})$$

(intersection inside $G(\mathbb{A})$) and consider

$$U_P \hookrightarrow G(\mathbb{R} \times \mathbb{Q}_P) \longrightarrow \Gamma \backslash G(\mathbb{R} \times \mathbb{Q}_P)$$

Let μ_P be the direct image of the Haar probability measure on U_P.
Let

$$\mu'_n = \mu_{U_P} \cdot g_n$$

We have

$$\pi_*(\mu'_n) = \mu_n$$

Very technical difficulties/details :

- WLOG we can assume $G = G^{der}$
- One needs to 'suitably modify the g_n'

The theorem of Richard-Zamojski now implies the equidistribution of the sequence $U_P \cdot g_n$ which impies the equidistribution theorem and which in turn implies the Topological and Zariski P-André-Pink-Zannier conjecture.

The main difficulty here is verifying the very technical conditions of their theorem.

Thank you for your attention !

Happy Birthday Umberto !