Algebraic actions of discrete groups.

This is a joint work with Cantat.

Let k be an algebraically closed field of char. 0.

Let X be an irreducible quasi-projective variety $/ k$

We want to study the following groups.

1. $\mathrm{Aut}(X)$ the group of automorphisms of X

2. $\mathrm{Bir}(X)$ the group of birational transformations of X.

A good example is provided by the affine spaces $\mathbb{A}^d$.

When $d=1$,

$\mathrm{Aut}(\mathbb{A}^1) = \{ az+b \mid a \in k^* \}$

$\mathrm{Bir}(\mathbb{A}^1) = \mathrm{Aut}(\mathbb{P}^1) = PGL_2(k)$

They are finitely dimensional algebraic groups.

When $d=2$, Jung's Thm $\Rightarrow$

$\mathrm{Aut}(\mathbb{A}^2)$ is generated by

$A = \{$ affine transformations $\}$

and

$E = \{$ elementary transformations $\}$

$$\begin{gathered} \| \\ (x,y) \longmapsto (x, y+P(x)) \end{gathered}$$

$P \in k[x]$.

Noether and Castelnuovo $\Rightarrow$

$\mathrm{Bir}(\mathbb{A}^2)$ is generated by

$PGL_3(k)$ and the Cremona involution

$\sigma: (x,y) \mapsto (\frac{1}{x}, \frac{1}{y})$.

These groups are "infinitely dimensional" and contains elements with a rich dynamical behavior, such as the Hénon mapping. Ex: $(x,y) \mapsto (y, x+y^2)$.

When $d \geq 3$, these groups are more complicated.

Recently, with Cantat, we present two new arguments to study these groups

One is based on p-adic analysis the other combines isoperimetric inequalities from geometric group theory with Lang-Weil estimates from diophantine geometry.

In particular, we prove the following result.

<u>Thm A</u> (Cantat, X.) Let Γ be a finite index subgroup of $SL_n(\mathbb{Z})$.
If $\exists$ embedding $\rho : \Gamma \hookrightarrow \mathrm{Bir}(X)$, then we have $\dim X \geq n-1$.

Remarks

1. If $\dim X = n-1$, then X is rational and the action of Γ on X is linear.

More precisely, $\exists$ birational map

$$\phi : X \dashrightarrow \mathbb{P}^{n-1}$$

such that $\phi \Gamma \phi^{-1} \subset \mathrm{Aut}(\mathbb{P}^{n-1})$

$$= PGL_n(k)$$

We note that if $\dim X > n-1$, this action may linear.

For example

$X = E^n$ where E is an elliptic curve.

$SL_n(\mathbb{Z}) \curvearrowright X$

This action is not linear in any birational model.

2. Thm A and Remark 1 also hold for $\mathrm{Aut}(X)$. In particular, if $\Gamma \subseteq \mathrm{Aut}(X)$ and $\dim X = n-1$, then $X \simeq \mathbb{P}^{n-1}$.

3. Thm A $\Longrightarrow$

1) $\mathrm{Aut}(\mathbb{A}^d) = \mathrm{Aut}(\mathbb{A}^{d'}) \Leftrightarrow d = d'$

2) $\mathrm{Bir}(\mathbb{A}^d) = \mathrm{Bir}(\mathbb{A}^{d'}) \Leftrightarrow d = d'$

3) $\dim X = d$ and $\mathrm{Bir}(X) = \mathrm{Bir}(\mathbb{P}^d)$

$\Longleftrightarrow X \overset{\mathrm{bir}}{\sim} \mathbb{P}^d$.

4. One can replace $SL_n(\mathbb{Z})$ by any non-cocompact lattice of any real Lie group S of $\mathrm{Rank}_{\mathbb{R}}(S) \geq 2$

in Thm A and we get $\dim X \geq \mathrm{Rank}_{\mathbb{R}} S$

5. In thm A, we may assume that $n \geq 3$. Then Γ is almost simple

: Its normal subgroups are either finite and central or co-finite.

Thus, the assumption "$\rho : \Gamma \hookrightarrow \mathrm{Bir}(X)$" can be replaced by "$\# \mathrm{Im}(\rho) = \infty$".

6. Thm A can be viewed as a birational version of <u>Zimmer conjecture</u>:

If M is a compact manifold with an embedding

$$\rho : \Gamma \hookrightarrow \underset{\{\text{diffeomorphisms}\}}{\underset{\shortparallel}{\mathrm{Diff}(M)}},$$

then $\dim M \geq n-1$.

7. When dim $X=2$, Thm A is proved by Déserti

Cantat has proved a version of it for the group of <u>automorphisms</u> of a <u>compact</u> Kähler manifold.

Generally speacking, the method of Cantat is to study the action of Γ on the cohomologie groups of X. In general, this method does not work for birational transformations and for automorphisms of non compact manifold.

However, in dim 2, Cantat has introduced an infinitely dimensional hyperbolic space on which Bir(X) acts by isometrics. The method of Déserti is to study the action of Bir(X) on ~~the~~ this space.

8. Our method apply also to actions of other discret groups, such as the mapping class groups or the nilpotent groups.

To prove Thm A, we need 2 steps.

1. We prove a p-adic local version of it.

2. Reduce to this local version.

1. We study the ~~p-adic~~ dynamics on the p-adic polydisk

$U = \mathbb{Z}_p^m$

Denote by $\mathrm{Diff}(U)$ the group of analytic automorphisms of U.

i.e. automorphism

$$f : U \longrightarrow U$$

$$(x_1, \cdots, x_m) \longmapsto (f_1, \cdots, f_m)$$

where $f_i = \sum_{I=(i_1,\cdots,i_m)} a_I X^I$

$a_I \in \mathbb{Z}_p$ and $|a_I| \to 0$

Recall

Thm 1 (Bell, Bell-Ghioca-Tucker, Poonen).

Let p be a prime number ≥ 3

Let f be an auto. in $\mathrm{Diff}(\mathcal{U})$

s.t. $f \equiv \mathrm{id} \mod p$

Then there exists an analytic map

$$\Phi : \mathbb{Z}_p \times \mathcal{U} \longrightarrow \mathcal{U}$$

s.t. $\Phi(n, x) = f^n(x) \quad \forall n \in \mathbb{Z}$.

Remark

1. We may view this thm like this.

If we have a morphism

$$(\mathbb{Z}, +) \hookrightarrow \mathrm{Diff}(\mathcal{U})$$
$$1 \longmapsto f$$

s.t. $f \equiv id \pmod{p}$

then we can extend it to a morphism

$$(\mathbb{Z}_p, +) \longrightarrow \mathrm{Diff}(\mathcal{U})$$

We see that

$\mathbb{Z}$ is a discrete group

but $\mathbb{Z}_p$ is a Lie group.

Then it induces a morphism of the Lie algebra

$$1 \longmapsto V_f := \left.\frac{\partial \Phi(t,x)}{\partial t}\right|_{t=0} \in \mathcal{H}(\mathcal{U})$$

$\swarrow$

it is a vector field on $\mathcal{U}$.

And $\Phi(t,x)$ is the flot induced by it.

With Cantat, we proved a non abelian version of Thm 1

<u>Thm 2</u> (Cantat, X.)

$p \geq 3$

Let Γ be a finite index subgroup of $Sl_n(\mathbb{Z})$

If $\exists$ embedding $\rho : \Gamma \hookrightarrow Diff(\mathcal{U})$

s.t. $\forall f \in \Gamma$, $\rho(f) \equiv id \pmod{p}$

then ρ extends to an analytic action of $\overline{\Gamma} \subset Sl_n(\mathbb{Z}_p)$ on $\mathcal{U}$.

Here $\overline{\Gamma}$ is the closure of Γ in $Sl_n(\mathbb{Z}_p)$ w.r.t. the p-adic topology. (It is open and closed.)

and we have an analytic map

$$\Phi : \overline{\Gamma} \times U \to U$$

s.t. $\Phi(g, x) = \rho(g)(x) \quad \forall g \in \Gamma, x \in U.$

(Here, we need the congruence subgroup property).

Then we can prove a local version of Thm A

Prop 3 (Cantat, X.).

$p \geq 3$

If $\exists$ embedding $\rho : \Gamma \hookrightarrow \mathrm{Diff}(U)$, $U = \mathbb{Z}_p^m$

s.t. $\rho(g) \equiv id \quad \forall g \in \Gamma$

then $m \geq n-1$.

Proof: By Thm 2, ρ extends to

$$\rho : \overline{\Gamma} \to \mathrm{Diff}(U).$$

By taking derivate, we get the tangent map

$$d\rho : \mathfrak{sl}_n(\mathbb{Q}_p) \longrightarrow \Theta(\mathcal{U})$$

Pick a general point $z_0 \in \mathcal{U}$, set

$$A := \{ v \in \mathfrak{sl}_n(\mathbb{Q}_p) \mid d\rho(v)\big|_{z_0} = 0 \}.$$

Then A is a proper Lie algebra of codimension at most m in $\mathfrak{sl}_n(\mathbb{Q}_p)$. On the other hand, the maximal proper Lie subalgebra of $\mathfrak{sl}_n(\mathbb{Q}_p)$ is of codimension $n-1$. So we have $n-1 \leq m$.

Now we prove thm A.

~~For the simplicity~~

Denote by $S = \{r_1, \cdots, r_s\}$

a finite symetric generating set of Γ.

For the simplicity, we suppose that all r_i and X are defined over $\mathbb{Q}$.

Pick an integral model

$$\begin{array}{c} \mathcal{X} \\ \downarrow \\ \operatorname{Spec}\mathbb{Z}. \end{array}$$

$\exists$ prime p.

s.t. X_p is irreducible.

$r_{i_p} = r_i \bmod p$ birational on X_p.

By base change, we assume that X is defined over $\mathbb{Q}_p$

Now if $\exists$ a p-adic polydisc $\mathcal{U} \subset X(\mathbb{Q}_p)$ which is invariant by Γ, then we may apply Thm2 to conclude the proof.

Finding such a polydisc is more or less finding a point $x \in X_p(\mathbb{F}_p)$ fixed by Γ i.e. $\forall \gamma \in \Gamma$, $x \notin I(\gamma_p)$ and $\gamma_p(x) = x$.

In fact, we don't really need that x is fixed by Γ, we only need that x is fixed by a finite index subgroup of Γ.

We also don't need ~~that~~ x ~~is~~ to be defined over $\mathbb{F}_p$, it could be defined over any finite extension of $\mathbb{F}_p$.

Then we may pose a general question.

<u>Q</u>: X_p irr. proj var. $/\overline{\mathbb{F}_p}$

$G \subset \mathrm{Bir}(X_p)$

finitely generated subgroup.

Can we find a point $x \in X_p(\overline{\mathbb{F}_p})$ and a finite index subgroup $G' < G$

s.t. x is fixed by G'.

Some times, we know the answer.

① If $G \subset \mathrm{Aut}(X_p)$, then Yes.

Because on a fixed field $\mathbb{F}_q$.

$X_p(\mathbb{F}_q)$ is finite.

② When $G = \mathbb{Z}$, Yes.

Twisted Lang-Weil estimate of Hrushorski;

③ When G has Kazhdan property (T), Yes.

Cantat-X. ~~using~~ $\Rightarrow$ Thm A.

because Γ has Kazhdan property (T)

When $G = \mathbb{Z}^2$, I don't know

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