Algebraic and holomorphic flows in the bi-algebraic context

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Hermitian Locally Symmetric Spaces-bi-Algebraic point of view.

The following transcendental maps relates "algebraic objects"

- $$\exp_g := (\exp, .., \exp) : \mathbb{C}^g \to (\mathbb{C}^*)^g. \tag{1}$$
- $$\pi : \mathbb{C}^g \to A(\mathbb{C}) = \Gamma \backslash \mathbb{C}^g \text{ with } A \text{ an abelian variety.} \tag{2}$$
- For $\mathcal{D}$ a bounded symmetric domain and Γ a torsion free lattice in $\mathrm{Aut}(\mathcal{D})$
$$\pi : \mathcal{D} \to S = \Gamma \backslash \mathcal{D}. \tag{3}$$

Hermitian Locally Symmetric Spaces-bi-Algebraic point of view.

- If Γ is an arithmetic lattice, $S = \Gamma\backslash\mathcal{D}$ is a quasi-projective variety (Baily-Borel).
- If Γ is irreducible of rank ≥ 2, Γ is arithmetic (Margulis).
- When Γ is of rank 1, so $\mathcal{D}$ is the unit ball in $\mathbb{C}^n$, Mok proved that the minimal compactification of S is projective.
- $\mathcal{D} \subset \mathfrak{p} = \mathbb{C}^n$ is semi-algebraic and complex analytic.

Bi-algebraic varieties and weakly special varieties.

Definition 1

- *Let $\pi : V \to W$ a transcendental map relating 2 algebraic objects. An algebraic subvariety Y of W is "bi-algebraic" if a component of $\pi^{-1}(Y)$ is algebraic.*
- *Let $\mathcal{D} \subset \mathfrak{p} = \mathbb{C}^n$ be a bounded symmetric domain. An irreducible algebraic subvariety Θ of $\mathcal{D}$ is a component of $\mathcal{D} \cap \tilde{\Theta}$ for an algebraic subvariety $\tilde{\Theta}$ of $\mathfrak{p}$. In this situation Θ is semi-algebraic and complex analytic.*

Proposition

A subvariety Y of an abelian variety A is bi-algebraic if and only if $Y = B + P$ for an abelian subvariety B of A and a point P. This is equivalent to saying that Y is a totally geodesic subvariety of A.

Bi-algebraic varieties and weakly special varieties.

Definition 2

- *Let* $S = \Gamma\backslash\mathcal{D}$ *be an hermitian locally symmetric space and* $\pi : \mathcal{D} \to S$ *be the uniformizing map. A special subvariety* S' *of* S *is a variety of the form* $S' = \Gamma'\backslash\mathcal{D}'$ *where* $\mathcal{D}'$ *is a bounded hermitian symmetric subspace of* $\mathcal{D}$ *and where* $\Gamma' := \Gamma \cap \mathrm{Aut}(\mathcal{D}')$ *is a lattice in* $\mathcal{D}'$*.*
- *A weakly special subvariety* V *of* S *is either special or there exists a special subvariety* $S' = S_1' \times S_2' = \Gamma_1'\backslash\mathcal{D}_1 \times \Gamma_2'\backslash\mathcal{D}_2$ *of* S *and a point* P *of* S_2 *such that* $V = S_1' \times \{P\} \subset S_1' \times S_2' = S'$*.*

Proposition 1 (U-Yafaev)

Assume that Γ *is arithmetic. A subvariety* V *of* $S = \Gamma\backslash\mathcal{D}$ *is bi-algebraic* $\iff$ V *is totally geodesic in* S $\iff$ V *is weakly special.*

Hyperbolic Ax-Lindemann conjecture.

Theorem 1 (Abelian Ax-Lindemann) Ax, Pila-Zannier

Let $\pi : \mathbb{C}^g \to A = \Gamma\backslash\mathbb{C}^g$ *and let* V *be a irreducible algebraic subvariety of* $\mathbb{C}^g$. *Then the Zariski closure* W *of* $\pi(V)$ *is bi-algebraic (i.e* $W = B + P$*).*

Theorem 2 (Hyperbolic Ax-Lindemann)

Assume that Γ *is an arithmetic lattice. Let* $\pi : \mathcal{D} \to S = \Gamma\backslash\mathcal{D}$ *and let* Y *be a irreducible algebraic subvariety of* $\mathbb{C}^g$. *Then the Zariski closure* V *of* $\pi(Y)$ *is weakly special (i.e totally geodesic or bi-algebraic).*

The proof is due to U-Yafaev when Γ is cocompact, Pila-Tsimerman for $\mathcal{A}_g$, Klingler-U-Yafaev for a general Shimura variety.

Bloch-Ochiai theorem

Theorem 3 (Bloch-Ochiai)

Let $\pi : \mathbb{C}^g \to A = \Gamma \backslash \mathbb{C}^g$*. Let* $f : \mathbb{C} \to \mathbb{C}^g$ *be a holomorphic map and* $V = f(\mathbb{C})$*. Then the Zariski closure* W *of* $\pi(V)$ *is bi-algebraic (i.e* $W = B + P$*).*

- The proof uses mainly Nevalinna theory.
- Generalization of this result and of Abelian Ax-Lidemann by Paun-Sibony for holomorphic maps from a subset of $\mathbb{C}$ to $\mathbb{C}^n$ with a growth estimate.
- For a unbounded real analytic subet, $V \subset \mathbb{C}^g$, definable in some o-minimal structure the Zariski closure of $\pi(V)$ is also bi-algebraic. (U-Yafaev).

Hyperbolic Bloch-Ochiai theorem

Theorem 4 (Hyperbolic Bloch-Ochiai) U-Yafaev

Let $\mathcal{D} \subset \mathfrak{p} = \mathbb{C}^g$ *be an hermitian bounded symmetric domain and* Γ *be an arithmetic and cocompact lattice of* $\mathcal{D}$. *Let* $\pi : \mathcal{D} \to S = \Gamma \backslash \mathcal{D}$. *Let* $f : \mathbb{C} \to \mathbb{C}^g$ *be a holomorphic map and* $V = f(\mathbb{C}) \cap \mathcal{D}$. *Then the connected components of the Zariski closure* W *of* $\pi(V)$ *are weakly special (i.e totally geodesic or bi-algebraic).*

- The proof uses mainly hyperbolic geometry, o-minimal theory, the hyperbolic Ax-Lindemann theorem and some of its consequences and a little bit of Nevanlinna theory. We don't k how to adapt this proof for the usual Bloch-Ochiai theorem.
- The case of $\mathcal{A}_g$ or of general Shimura varieties is open.

What about the topological closure ?

Let $\pi : X \longrightarrow Y$ be a transcendental map between two "algebraic objects". We saw several natural examples of such maps and subsets Θ of X such that the Zariski closure of $\pi(\Theta)$ in Y, is bi-algebraic.

Question

In this situation, what can be said about the topological closure $\overline{\pi(\Theta)}$ of $\pi(\Theta)$?

Real weakly special subvarieties

Let $A = \mathbb{C}^g/\Gamma$ be a complex abelian variety.

Definition

Let $W \subset \mathbb{C}^g$ be a $\mathbb{R}$-vector space such that $\Gamma_W := \Gamma \cap W$ is a lattice in W. Then W/Γ_W is a real torus and is a closed real analytic subset of A. A real analytic subvariety V of A is said to be **real weakly special** *if $V = P + W/\Gamma_W$ for a point P and a real subtorus W/Γ_W of A.*

Mumford-Tate tori

Definition

Let Θ be an irreducible algebraic subvariety of $\mathbb{C}^g$ containing the origin O of $\mathbb{C}^g$. The **Mumford-Tate group** *$MT(\Theta)$ of Θ is defined as the smallest $\mathbb{Q}$-vector subspace W of $\Gamma \otimes \mathbb{Q}$ such that $\Theta \subset W \otimes \mathbb{R}$. More generally, if $P \in \Theta$. Then we define $MT(\Theta)$ as $\mathrm{MT}(\Theta - P)$. One can check that the definition is independent of the choice of $P \in \Theta$. Let $W_\Theta := MT(\Theta) \otimes \mathbb{R}$. We denote by $\mathbb{T}_\Theta$ the real weakly-special subvariety of A*

$$\mathbb{T}_\Theta = \pi(P) + W_\Theta / \Gamma \cap W_\Theta .$$

Then $\mathbb{T}_\Theta$ is independent of P and $\mathbb{T}_\Theta$ is the smallest real weakly special subvariety of A containing $\pi(\Theta)$. We say that $\mathbb{T}_\Theta$ is the **Mumford-Tate torus** *associated to Θ. We write μ_Θ for $\mu_{\mathbb{T}_\Theta}$.*

Remark

Let Θ be an irreducible complex algebraic subvariety of $\mathbb{C}^g$. Then $\overline{\pi(\Theta)} \subset \mathbb{T}_\Theta$. When do we have $\overline{\pi(\Theta)} = \mathbb{T}_\Theta$?

Asymptotic Mumford-Tate tori.

Let C be a curve in $\mathbb{C}^g$. Let C^* be the Zariski closure of C in $\mathbb{P}^1(\mathbb{C})^g$. Then $C^* - C$ is a finite set of points $\{P_1, \ldots, P_s\}$. Let C_α be a branch of C near a point P_i. There exists a smallest real affine subspace $Q_\alpha + W_\alpha$ such that $W_\alpha \cap \Gamma$ is a lattice in W_α and such that C_α is asymptotic to $Q_\alpha + W_\alpha$.

Definition

Let $\mathbb{T}'_\alpha := W_\alpha/\Gamma \cap W_\alpha$ *and* $\mathbb{T}_\alpha := \pi(Q_\alpha) + \mathbb{T}'_\alpha$. *We say that* $\mathbb{T}_\alpha$ *is the* **asymptotic Mumford-Tate torus** *associated to* C_α

Topological closure of an algebraic flow.

Theorem 5 (U-Yafaev)

Let C be a curve in $\mathbb{C}^g$. Let $C_1, \dots, C_r$ be the set of all **branches** *of C through all points at infinity. For all $\alpha \in \{1, \dots, r\}$ let $\mathbb{T}_\alpha$ be the associated* **asymptotic Mumford-Tate torus**. *Then*

$$\overline{\pi(C)} = \pi(C) \cup \bigcup_{\alpha=1}^{r} \mathbb{T}_\alpha .$$

The theorem has a version in terms of measures. The proof uses the Weyl criterion, explicit computations of the character groups of the asymptotic Mumford-Tate tori and harmonic analysis in particular some results on oscillatory integrals (Van der Corput lemma).

Example : The linear case.

The case of W a complex linear subspace of $\mathbb{C}^g$ is a simple application of Weyl's criterion. In this case $\overline{\pi(W)} = \mathbb{T}_W$ and $\mu_{Z,R} \to \mu_W$.

Real tori are needed

Let V be a $\mathbb{C}$-vector space of dimension 2 and (e_1, e_2) be a $\mathbb{C}$-basis of V. Let Γ be the lattice

$$\Gamma := \mathbb{Z}e_1 \oplus \mathbb{Z}\sqrt{-1}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}\sqrt{-5}e_2$$

Then $A := A/\Gamma$ is an abelian variety of dimension 2. Let $W := \mathbb{C}(e_1 + e_2)$ of V.

$$MT(W) = \mathbb{Q}(e_1 + e_2) + \mathbb{Q}\sqrt{-1}e_1 + \mathbb{Q}\sqrt{-5}e_2$$

and

$$MT(W) \otimes \mathbb{R} = \mathbb{R}(e_1 + e_2) + \mathbb{R}\sqrt{-1}e_1 + \mathbb{R}\sqrt{-5}e_2.$$

As a consequence $MT(W) \otimes \mathbb{R}/\Gamma \cap MT(W) \otimes \mathbb{R}$ is a real torus of real dimension 3.

This shows that we can't expect that in the conjecture 2 that the analytic closure of $\pi(W)$ has a complex structure.

Instructing example 1

Proposition

Let $n \geq 3$ be an integer. Let $C \in \mathbb{C}^2$ be the hyperelliptic curve with equation

$$Z_2^2 = Z_1^n + a_{n-1} Z_1^{n-1} + \cdots + a_0.$$

Then for any abelian surface $A = \mathbb{C}^2/\Gamma$ we have $\overline{\pi(C)} = A$ and $\mu_{C,R} \to \mu_A$ as $R \to \infty$. In this case $\mathbb{T}_C = A = \mathbb{T}_\alpha$ for all infinite branches C_α of C.

Instructing example 2

Let C be the hyperbole $Z_1 Z_2 = 1$ in $\mathbb{C}^2$.

case 1.

Let $\Gamma = \mathbb{Z}[\sqrt{-1}] \oplus \mathbb{Z}[\sqrt{-1}] \subset \mathbb{C}^2$ and $A = E \times E = \mathbb{C}^2/\Gamma$. Then

$$\overline{\pi(C)} = \pi(C) \cup E \times \{0\} \cup \{0\} \times E,$$

and

$$\mu_{C,R} \to \frac{1}{2}(\mu_{E\times\{0\}} + \mu_{\{0\}\times E}).$$

In this case $\mathbb{T}_C = A$, with two branches C_1 near $(0,\infty)$ and C_2 near $(\infty, 0)$. Then $\mathbb{T}_1 = \{0\} \times E$ and $\mathbb{T}_2 = E \times \{0\}$.

case 2.

If $\Gamma \subset \mathbb{C}^2$ is such that the dual lattice $\widehat{\Gamma}$ of Γ contains no element of the form $(0, b)$ or of the form $(a, 0)$, then $\overline{\pi(C)} = \mathbb{C}^2/\Gamma = A$ and $\mu_{C,R} \to \mu_A$. In this case $\mathbb{T}_C = \mathbb{T}_1 = \mathbb{T}_2 = A$.

The results of Peterzil-Starchenko

Theorem 6 (Peterzil-Starchenko)

Let Θ be a algebraic subvariety of $\mathbb{C}^g$. There exists finitely many algebraic subvarieties $C_1, \dots, C_m$ of $\mathbb{C}^g$, finitely many complex vector subspaces $V_1, \dots, V_m$ depending only on Θ such that

$$\overline{\pi(\Theta)} = \pi(\Theta) \cup \cup_{i=1}^{m} (\pi(C_i) + \mathbb{T}_i)$$

where $\mathbb{T}_i = \mathbb{T}_{V_i}$ is the Mumford-Tate torus of V_i.
Moreover $\dim(C_i) < \dim(\Theta)$.

Remark

- *If $\dim(\Theta) = 1$ they give a new proof of the theorem 5.*
- *C_I and V_i are independent of Γ but $\mathbb{T}_i = \mathbb{T}_{V_i}$ depends on Γ.*
- *$\pi(\Theta)$ and $\pi(C_i)$ are in general neither closed nor definable in a o-minimal structure.*
- *I don't know what should be the measure theoretic version of the theorem*
- *The proof uses many inputs from Model theory and o-minimal theory.*

Algebraic flows on hermitian locally symmetric spaces

Let $\pi : \mathcal{D} \longrightarrow S = \Gamma\backslash\mathcal{D}$ be the uniformizing map of a Shimura variety.

Definition

A **real weakly special subvariety** *of S is a real analytic subset of S of the form*

$$Z = \Gamma \cap H(\mathbb{R})^+ \backslash H(\mathbb{R})^+ .x$$

where H is an algebraic subgroup of G such that the radical of H is unipotent and the real points of the $\mathbb{Q}$-simple factors of a Levi of H are not compact and $x \in \mathcal{D}$.

Theorem

Let V be a complex totally geodesic subspace of $\mathcal{D}$. Then $\overline{\pi(V)}$ is real weakly special.

Proof : ergodic theory (Ratner's theorem).

Example

Let $G = \mathrm{SL}_2 \times \mathrm{SL}_2$, $X^+ = \mathbb{H} \times \mathbb{H}$ *and for some* $g \in \mathrm{SL}_2(\mathbb{R})$

$$Z = \{(\tau, g\tau), \tau \in \mathbb{H}\}.$$

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z}) \times \mathrm{SL}_2(\mathbb{Z})$ *and*

$$\pi : \mathbb{H} \times \mathbb{H} \longrightarrow \Gamma \backslash X^+ = Y_0(1) \times Y_0(1).$$

- *If* $g \in G(\mathbb{Q})$, *the closure of* $\pi(Z)$ *is a special subvariety* $Y_0(n)$ *for some* $n \in \mathbb{N}$.
- *If* $g \notin G(\mathbb{Q})$, *then* $\pi(Z)$ *is dense in* $\Gamma \backslash X^+$.

Question

Question

Let Θ *be an algebraic subvariety of* X^+*. Describe* $\overline{\pi(\Theta)} \subset S$ *in term of real weakly special subvarieties.*

Example

Let C' *be an algebraic curve in* $\mathbb{C}^2$*. Let* $C = C' \cap \mathbb{H} \times \mathbb{H}$*. Let*

$$\pi = j \times j : \mathbb{H} \times \mathbb{H} \longrightarrow Y_0(1) \times Y_0(1).$$

Describe $\overline{\pi(C)}$*.*

THANKS TO THE ORGANIZERS

HAPPY BIRTHDAY UMBERTO