Rationality problems

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(R) rational: if $X \sim \mathbb{P}^n$ for some n

(S) stably rational: if $X \times \mathbb{P}^n$ is rational, for some n

(U) unirational: if $\mathbb{P}^n \dashrightarrow X$, for some n

Classical results

In dimensions ≤ 2, over $\mathbb{C}$,

$$\text{rationality} = \text{stable rationality} = \text{unirationality}$$

- Curves: Lüroth
- Surfaces: Castelnuovo, Enriques

This can fail over nonclosed ground-fields k.

Del Pezzo surfaces over nonclosed fields

Theorem

Let X be a smooth del Pezzo surface over a field k.

- $\deg(X) \geq 5$: *If $X(k) \neq \emptyset$ then X is k-rational.*
- $\deg(X) = 4, 3, (2)$: *If $X(k) \neq \emptyset$ then X is k-unirational.*

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- $\deg(X) = 4, 3, (2)$*: If* $X(k) \neq \emptyset$ *then* X *is* k*-unirational.*
- $\deg(X) = 1$*:* $X(k) \neq \emptyset$*. Is* $X(k)$ *Zariski dense? Is* X *unirational? (Some results by Salgado and van Luijk, 2014.)*

Cohomology

Let

$$\mathrm{H}^i(G, M)$$

be the i-cohomology group of a finite or profinite group G, with coefficients in a G-module M. Recall:

- $\mathrm{H}^0(G, M) = M^G$, the submodule of G-invariants

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Obstruction to rationality

$$\mathrm{Br}(X) = \mathrm{H}^2_{et}(X, \mathbb{G}_m).$$

For Del Pezzo surfaces,

$$\mathrm{Br}(X)/\mathrm{Br}(k) = \mathrm{H}^1(G_k, \mathrm{Pic}(\bar{X})).$$

Let $X \subset \mathbb{P}^4$ be a smooth DP4. The Galois action on the 16 lines factors through the Weyl group $\mathsf{W}(\mathsf{D}_5)$ (a group of order 1920).

Bright, Bruin, Flynn, Logan 2007

- If the degree of the splitting field over $\mathbb{Q}$ is > 96 then

$$\mathrm{H}^1(G_{\mathbb{Q}}, \mathrm{Pic}(\bar{X})) = 0.$$

Computing the obstruction group

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- If the degree of the splitting field over $\mathbb{Q}$ is > 96 then
$$\mathrm{H}^1(G_{\mathbb{Q}}, \mathrm{Pic}(\bar{X})) = 0.$$
- In all other cases, the obstruction group is either
$$1, \mathbb{Z}/2\mathbb{Z}, \text{ or } (\mathbb{Z}/2\mathbb{Z})^2.$$

The obstruction group

This obstruction is effectively computable for all Del Pezzo surfaces over number fields.

Obstruction to stable rationality

If X is stably rational then $\mathrm{H}^1(G_{k'}, \mathrm{Pic}(\bar{X})) = 0$, for all k'/k.

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Conjecture (Colliot-Thélène–Sansuc)

If $X(k) \neq \emptyset$ and this obstruction vanishes then X is stably rational.

The only known case:

Example

Let X be a conic bundle over $\mathbb{P}^1$, over a field k, given by

$$x^2 - ay^2 = f(s)z^2, \quad \deg(f) = 3, \quad \operatorname{disc}(f) = a,$$

with f irreducible over k. Then X is nonrational over k, but

$$\mathrm{H}^1(G_{k'}, \mathrm{Pic}(\bar{X})) = 0, \qquad \text{for all } k'/k.$$

Beauville–Colliot-Thélène–Sansuc–Swinnerton-Dyer 1985:

X is stably rational

Candidates, DP4:

- I_1: $y^2 - xz^2 = (x-3)(x+3)(x^3+9)$
- I_2: $y^2 - xz^2 = -(x^3 + 2apx^2 + a^2p^2x - a^3q^3)(x^2 - 2rx + s)$, such that
  - a is not a cube,
  - $g(x) := x^3 + px + q$ is irreducible,
  - $\mathrm{disc}(g)/(r^2-s)$, $s/(r^2-s)$, and $a/\mathrm{disc}(g)$ are squares
- I_3: $y^2 - xz^2 = -(x^2-3)(x^3+3)$

Data:

- G/k linear algebraic group (e.g., finite group)
- $\rho : G \to V$ faithful representation

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Noether's problem

Is $X := V/G$ rational?

More generally, G acting on a variety Y, is $X := Y/G$, resp. $k(Y)^G$, rational?

Why interesting? Applications to the inverse problem of Galois theory - realizing a finite group G as the Galois group of a field extension (via Hilbert's irreducibility).

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Why difficult? $\mathrm{Gr}(2,n) = \mathrm{SL}_2 \backslash \mathrm{Mat}_{2\times n}$ is rational. The ring of invariants has $\binom{n}{2}$ generators and $\binom{n}{4}$ relations.

- If G is SL_n, Sp_n, SO_n, ... then V/G is stably rational.
- If $G = \mathrm{PGL}_3$ then V/G is rational (Böhning–von Bothmer 2008)

Noether's problem: counterexamples

Nonlinear actions: Saltman (1984)

Let

- $G = (\mathbb{Z}/p)^3$, p prime,
- $M := \mathrm{Ker}(\mathbb{Z}[G \times G] \to \mathbb{Z}[G])$,
- $X = \mathrm{Spec}(k[M])$,

Then X/G is not rational.

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Linear actions: Bogomolov (1988)

Nontriviality of the unramified Brauer group of the function field $k(V)^G$, for some group of order p^6. In particular, V/G is not stably rational.

- Obstruction lies in Galois cohomology $\mathrm{H}^2_{nr}(k(V/G), \mathbb{Z}/\ell)$,

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- Starting point of birational Almost abelian anabelian geometry program of Bogomolov

Unramified cohomology and the Brauer group

Let $K = k(X)$ be a function field over $k = \bar{k}$, $G_K := \mathrm{Gal}(\bar{K}/K)$ its Galois group, and

$$\mathrm{H}^i(K) := \mathrm{H}^i(G_K, \mathbb{Z}/n)$$

its i-th Galois cohomology. For every divisorial valuation ν of K we have a natural homomorphism

$$\mathrm{H}^i(K) \xrightarrow{\partial_\nu} \mathrm{H}^{i-1}(\kappa(\nu))$$

The group

$$\mathrm{H}^i_{nr}(K) := \cap_\nu \mathrm{Ker}(\partial_\nu)$$

is a birational invariant; it vanishes for rational K. For smooth X we have

$$\mathrm{H}^2_{nr}(K) = \mathrm{Br}(X)[n]$$

Theorem (Bogomolov–T. 2015)

Let X be a variety of dimension ≥ 2 over $k = \bar{\mathbb{F}}_p$, $K = k(X)$, and $\ell \neq p$. Every $\alpha \in \mathrm{H}^i_{nr}(K, \mathbb{Z}/\ell))$ is induced from an unramified class in the cohomology of a quotient

$$(\prod_j \mathbb{P}(V_j))/G^a,$$

for some finite ℓ-group G^a.

Threefolds

The Minimal Model Program implies that rationally connected 3-folds are of three types:

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- Del Pezzo fibrations over $\mathbb{P}^1$
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- Conic bundles over a rational surface.

Many (all??) of these are unirational.

Lüroth's problem

Does unirationality imply rationality?

There were numerous unsuccessful attempts to find counterexamples.

Osservazioni sopra alcune varietà non razionali aventi tutti i generi nulli.

di GINO FANO.

In un lavoro pubblicato alcuni anni or sono negli " Atti „ di questa R. Accademia (1) ho dimostrato che la varietà del 4° ordine dello spazio S_4 priva di punti doppi, e la varietà M_3^6 di S_5 intersezione generale di una quadrica e di una varietà cubica di quest'ultimo spazio, pur avendo tutti i generi nulli, non sono razionali. La dimostrazione era fondata sull'impossibilità di soddisfare in pari tempo a certe condizioni, tutte necessarie per l'esistenza di sistemi omaloidici di superficie contenuti rispett. in quelle due varietà.

Counterexamples to Lüroth's problem

Major developments in 1971-72:

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Major developments in 1971-72:

- Iskovskikh-Manin: quartic in $\mathbb{P}^4$ via birational rigidity
- Clemens-Griffiths: cubic in $\mathbb{P}^4$ via intermediate Jacobians
- Artin-Mumford: conic bundles via unramified cohomology

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- Reid, Pukhlikov, Cheltsov: birational rigidity of many smooth and singular (high degree) Fano hypersurfaces in weighted projective spaces
- Some of these are known to be unirational. **Guess:** a (very general) birationally rigid threefold is not stably rational.

Theorem

If the intermediate Jacobian $\mathrm{IJ}(X)$ *of a threefold* X *is not a product of Jacobians of curves then* X *is nonrational.*

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Implementation:

- Cubic threefolds (Clemens–Griffiths)
- Intersection of 3 quadrics and conic bundles (Beauville)
- Del Pezzo surface fibrations over $\mathbb{P}^1$ (Alexeev, Kanev, Grinenko, Cheltsov)

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Limitation: Does not detect failure of stable rationality

Idea (Clemens 1974): Let

$$\phi : \mathcal{X} \to B$$

be a family of Fano threefolds, with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber

$$X := \phi^{-1}(b)$$

satisfies the following conditions:

(S) Singularities: X has at most rational double points

(O) Obstruction: the intermediate Jacobian $\mathrm{IJ}(\tilde{\mathcal{X}}_0)$ (of the resolution of singularities $\tilde{\mathcal{X}}_0$) is not a product of Jacobians of curves.

Then a general fiber $\mathcal{X}_b$ is not rational.

Specialization method

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Implementation (Beauville 1977): nonrationality of certain Fano varieties

Theorem (Artin-Mumford)

Let $X \to S$ be a conic bundle over a smooth projective rational surface with discriminant a smooth curve

$$D = \sqcup_{j=1}^{r} D_j \subset S,$$

and with $g(D_j) \geq 1$ for all j. Then

$$\mathrm{H}^2_{nr}(k(X), \mathbb{Z}/2) = (\mathbb{Z}/2)^{r-1}.$$

Unramified cohomology

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Implementation: A special conic bundle over $\mathbb{P}^2$.

Cycle-theoretic tools: CH_0

$\mathrm{CH}_0(X_k)$ is the abelian group generated by zero-dimensional subvarieties $x \in X$ (e.g., points $x \in X(k)$), modulo k-rational equivalence.

Assuming $X(k) \neq \emptyset$, there is a surjective homomorphism

$$\deg : \mathrm{CH}_0(X_k) \to \mathbb{Z}.$$

For which X is this an isomorphism?

Example

- X a unirational or rationally-connected variety over $k = \mathbb{C}$.

CH_0-triviality

A projective X/k is universally CH_0-trivial if for all k'/k

$$\mathrm{CH}_0(X_{k'}) \xrightarrow{\sim} \mathbb{Z}$$

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For example, smooth k-rational varieties are universally CH_0-trivial. Unirational or rationally-connected varieties are not necessarily universally CH_0-trivial. Varieties with nontrivial unramified cohomology groups are not universally CH_0-trivial.

CH_0-TRIVIALITY

This condition is difficult to check, in general. Here is a sample of results: Universal CH_0-triviality holds for

- For cubic threefolds parametrized by a countable union of subvarieties of codimension ≥ 3 of the moduli space (Voisin 2014); these should be dense in moduli
- For special cubic fourfolds with discriminant not divisible by 4 (Voisin 2014)
- For cubic fourfolds (of discriminant 8) containing a plane (Auel–Colliot-Thélène–Parimala, 2015)

Specialization method
Voisin 2014, Colliot-Thélène–Pirutka 2015

Let

$$\phi : \mathcal{X} \to B$$

be a flat projective morphism of complex varieties with smooth generic fiber. Assume that there exists a point $b \in B$ such that the fiber

$$X := \phi^{-1}(b)$$

satisfies the following conditions:

(S) Singularities: X has mild singularities

(O) Obstruction: the group $\mathrm{H}^2_{nr}(\mathbb{C}(X), \mathbb{Z}/2)$ is nontrivial.

Then a very general fiber of ϕ is not stably rational.

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Still in development: new version by Schreieder (2017).

Specialization method: First applications

Very general varieties below are not stably rational:

- Quartic double solids $X \to \mathbb{P}^3$ with ≤ 7 double points (Voisin 2014)
- Quartic threefolds (Colliot-Thélène–Pirutka 2014)
- Sextic double solids $X \to \mathbb{P}^3$ (Beauville 2014)
- Fano hypersurfaces of high degree (Totaro 2015)
- Cyclic covers $X \to \mathbb{P}^n$ of prime degree (Colliot-Thélène–Pirutka 2015)
- Cyclic covers $X \to \mathbb{P}^n$ of arbitrary degree (Okada 2016)

Conic bundles over rational surfaces

Theorem (Hassett-Kresch-T. 2015)

A very general conic bundle $X \to S$, over a rational surface S, with discriminant of sufficiently high degree, e.g., $X \to \mathbb{P}^2$, with discriminant a curve of degree ≥ 6, is not stably rational.

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Theorem (Kresch-T. 2017)

Similar result for 2-dimensional Brauer-Severi bundles over rational surfaces.

Conic bundles over higher-dimensional bases

Stable rationality fails for general varieties in the following families:

- Certain conic bundles over $\mathbb{P}^3$, e.g.,

  $$X \subset \mathbb{P}^2 \times \mathbb{P}^3$$

  of bi-degree $(2,2)$ (Auel–Böhning–von Bothmer–Pirutka 2016)
- Conic bundles over $\mathbb{P}^{n-1}$: smooth $X \subset \mathbb{P}(\mathcal{E})$, for $\mathcal{E}$ direct sum of three line bundles, if $-K_X$ is not ample. In particular

  $$X \subset \mathbb{P}^2 \times \mathbb{P}^{n-1}$$

  of bi-degree $(2,d)$, $d \geq n \geq 3$ (Ahmadinezhad–Okada 2017)

Let $X \to S$ be a very general conic bundle over a del Pezzo surface of degree 1, with discriminant $C \in |-2K_S|$. Then

- X is not birationally rigid
- $IJ(X)$ is an elliptic curve
- X has trivial Brauer group

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Del Pezzo fibrations

Theorem (Hassett-T. 2016)

A very general fibration $\pi : \mathcal{X} \to \mathbb{P}^1$ in quartic del Pezzo surfaces which is not rational and not birational to a cubic threefold is not stably rational.

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Theorem (Krylov-Okada 2017)

A very general del Pezzo fibration $\pi : \mathcal{X} \to \mathbb{P}^1$ of degree 1, 2, or 3 which is not rational and not birational to a cubic threefold is not stably rational.

Fano threefolds

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Generalizations by Okada to certain singular Fano varieties.

Find suitable degenerations with mild singularities and birational to conic bundles.

Nonrational Fano threefolds with

$$\mathrm{Pic}(V) = -K_V\mathbb{Z} \quad \text{and} \quad d = d(V) = -K_V^3 :$$

- $d = 2$ sextic double solid
- $d = 4$ quartic
- $d = 6$ intersection of a quadric and a cubic
- $d = 8$ intersection of three quadrics
- $d = 10$ section of $\mathrm{Gr}(2,5)$ by two linear forms and a quadric
- $d = 14$ birational to a cubic threefold

Fano threefolds: degenerations

From general quartic del Pezzo $\mathcal{X} \to \mathbb{P}^1$ to Fano threefolds V:

- $d = 2$: $h(\mathcal{X}) = 22 \Rightarrow$ sextic double solid V with 32+4 nodes
- $d = 4$: $h(\mathcal{X}) = 20 \Rightarrow$ quartic threefold with 16 nodes
- $d = 6$: $h(\mathcal{X}) = 18 \Rightarrow$ quadric $\cap$ cubic with 8 nodes
- $d = 8$: $h(\mathcal{X}) = 16 \Rightarrow$ intersection of three quadrics with 4 nodes
- $d = 10$: $h(\mathcal{X}) = 14 \Rightarrow$ specialization of a V with 2 nodes

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The other families of Fano threefolds are conic bundles, but not very general, as in the theorem above. Additional work is needed.

Fano threefolds and del Pezzo fibrations

Consider the intersection of two $(1,2)$-hypersurfaces in $\mathbb{P}^1 \times \mathbb{P}^4$:

$$sP_1 + tQ_1 = sP_2 + tQ_2 = 0.$$

Let $v_1, \dots, v_{16} \in \mathbb{P}^4$ denote the solutions to

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- Projection onto the first factor gives a degree 4 del Pezzo fibration over $\mathbb{P}^1$ (with 16 constant sections)
- Projection onto the second factor gives a quartic threefold

  $$V := \{P_1 Q_2 - Q_1 P_2 = 0\} \subset \mathbb{P}^4$$

  with 16 nodes $v_1, \ldots, v_{16}$.

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Example

$X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, double cover ramified in a $(2, 2, 2)$ hypersurface; conic bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ with discriminant of bi-degree $(4, 4)$ – not generic in its linear series!

Fano threefolds of higher Picard rank

The other families of Fano threefolds are conic bundles, but not very general, as in the theorem above. Additional work is needed.

Example

$X \to \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$, double cover ramified in a $(2, 2, 2)$ hypersurface; conic bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ with discriminant of bi-degree $(4, 4)$ – not generic in its linear series! The corresponding K3 double cover $S \to \mathbb{P}^1 \times \mathbb{P}^1$ has Picard rank 3 and not 2.

Rationality in families

Let $\pi : \mathcal{X} \to B$ be a family of rationally connected varieties and put

$$\mathrm{Rat}(\pi) := \{\, b \in B \mid \mathcal{X}_b \quad \text{is rational} \,\}.$$

de Fernex–Fusi 2013

In dimension 3, $\mathrm{Rat}(\pi)$ is a countable union of closed subsets of B.

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Remark

Over number fields, $\mathrm{Rat}(\pi)$ has been studied, in connection with specializations in Brauer-Severi fibrations (Serre's problem).

Rat(π): Hassett-Pirutka-T. 2016

Rat(π) and its complement can be dense on the base.

There exist smooth families of projective rationally connected fourfolds $\mathcal{X} \to B$ over $k = \mathbb{C}$ such that:

- For every $b \in B$ the fiber X_b is a quadric surface bundle over a rational surface S;
- For very general $b \in B$ the fiber $\mathcal{X}_b$ is not stably rational;
- The set of $b \in B$ such that $\mathcal{X}_b$ is rational is dense in B.

Two difficulties:

- Construction of special X satisfying (**O**) and (**S**)

Rat(π): Hassett-Pirutka-T. 2016

Rat(π) and its complement can be dense on the base.

There exist smooth families of projective rationally connected fourfolds $\mathcal{X} \to B$ over $k = \mathbb{C}$ such that:

- For every $b \in B$ the fiber X_b is a quadric surface bundle over a rational surface S;
- For very general $b \in B$ the fiber $\mathcal{X}_b$ is not stably rational;
- The set of $b \in B$ such that $\mathcal{X}_b$ is rational is dense in B.

Two difficulties:

- Construction of special X satisfying (**O**) and (**S**)
- Rationality constructions

Rationality in families: idea

Consider a quadric surface bundle

$$\pi : \mathcal{Q} \to \mathbb{P}^2,$$

with smooth generic fiber. Let $D \subset \mathbb{P}^2$ be the degeneration curve; assume that D is smooth. Then $\mathcal{Q}$ is characterized by:

- the double cover $T \to \mathbb{P}^2$ with ramification in D
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When $\deg(D) \geq 6$, $\mathrm{Pic}(T)$ and $\mathrm{Br}(T)$ can change as we vary D.

Rationality in families: implementation

We consider bi-degree $(2,2)$ hypersurfaces

$$X \subset \mathbb{P}^2 \times \mathbb{P}^3.$$

Projection onto the first factor gives a quadric bundle over $\mathbb{P}^2$, its degeneration divisor $D \subset \mathbb{P}^2$ is an octic curve.

Let

$$X \subset \mathbb{P}^2_{[x:y:z]} \times \mathbb{P}^3_{[s:t:u:v]}$$

be a bi-degree $(2,2)$ hypersurface given by

$$yzs^2 + xzt^2 + xyu^2 + F(x,y,z)v^2 = 0,$$

where

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The discriminant curve for the projection $X \to \mathbb{P}^2$ is given by

$$x^2y^2z^2F(x,y,z) = 0.$$

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Special fiber

- Computing $\mathrm{H}^2_{nr}(\mathbb{C}(X), \mathbb{Z}/2)$: general approach by Pirutka (2016)
- Desingularization: by hand; the singular locus is a union of 6 conics, intersecting transversally

Rationality

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- Then the quadric over the function field $\mathbb{C}(\mathbb{P}^2)$ has a point, and X is rational.
- The corresponding locus is dense in the usual topology of the moduli space.

Other applications: Hassett–Pirutka–T. 2017

Let $X \subset \mathbb{P}^7$ be a very general intersection of three quadrics. Then X is not stably rational. Rational X are dense in moduli.

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Idea: Such X admit a fibration $X \to \mathbb{P}^2$, with generic fiber a quadric surface and octic discriminant.

Smooth cubic hypersurfaces $X_3 \subset \mathbb{P}^n$

- dim $= 1$ - nonrational

SMOOTH CUBIC HYPERSURFACES $X_3 \subset \mathbb{P}^n$

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- dim = 4 - periodicity??

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Unirational parametrizations:

- all admit unirational parametrizations of degree 2
- (Hassett-T. 2001) Cubic fourfolds with an odd degree unirational parametrization are dense in moduli

Special cubic fourfolds

Addington–Hassett–T.–Várilly-Alvarado 2016

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Idea: Every $X \in \mathcal{C}_{18}$ admits a fibration $X \to \mathbb{P}^2$ with general fiber a degree 6 Del Pezzo surface. A multisection of degree coprime to 3 forces rationality. The locus of such cubics is dense in $\mathcal{C}_{18}$.

Remark

Something like this should work for 6-dimensional cubics.

Summary

- The specialization method of Voisin, further developed by Colliot-Thélène–Pirutka, has triggered new advances in the study of rationality properties of higher-dimensional varieties.

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- The specialization method of Voisin, further developed by Colliot-Thélène–Pirutka, has triggered new advances in the study of rationality properties of higher-dimensional varieties.
- Stable rationality of general threefolds is essentially settled.
- Rationality properties can change in smooth families in dimension ≥ 4.
- Rationality and stable rationality of cubics remain a challenge.