Self maps of $\mathbb{P}^1$ with fixed degeneracies

Lucien Szpiro

We introduce *differential good reduction* for self maps of $\mathbb{P}^1_K$ and prove a Shafarevich type finiteness theorem:

Theorem

Let $d \geqslant 2$ be an integer and let K be a number field or a function field over an algebraically closed field k or a finite field k of characteristic $p > 2d - 2$. The set of $\mathrm{PGL}(2)$ isomorphism classes of non-isotrivial, self maps of $\mathbb{P}^1$ of degree d, ramified in at least 3 points, with differential good reduction outside a given finite set S of places of K, is finite.

Joint work with Tom Tucker and Lloyd West.

Outline

The talk will have three parts:

I- Arithmetico-Geometric Shafarevich "conjectures", theorems and counterexamples

II- Notions of good reduction for self maps of $\mathbb{P}^1$: simple good reduction , critical good reduction, differential good reduction

III- Proof of the finiteness theorem using the S-unit theorem and Mori-Grothendiek's tangent map to the scheme parametrising maps of schemes with prescribe behavior on a closed subsheme

Arithmetico-Geometric Shafarevich conjectures, theorems and counterexamples: I. Generalities

We fix K a number field or a functions field of a smooth connected curve C over a field k, and a finite set S of places of K. We give a geometric object X over K with a set of properties P.

Define $\text{Ш}(P, K, S)$ to be the set of X with the proprieties P and, for every place $v \notin S$ a model over $\mathcal{O}_v$, whose reduction mod v has the same properties P up to automorphisms.

The general question is finiteness of such a set. It is not always true but many cases of finiteness have been proved.

Shafarevich theorems: II. Finite maps

If we take $Ш(N, K, S)$ to be the set of finite coverings of degree N ramified only over S, we get a finite set in the following cases:

(i) K a number field (Minkowski, Hensel)
(ii) K a function field of one variable of charscteristic zero and $|S| \geqslant 3$ (Riemann)
(iii) K a function field of char $p > 0$, tame ramification and $|S| \geqslant 3$

But $Ш(N, K, S)$ is not finite for reason of wild ramification for:

(iv) K a function field of one variable of characteristic $p > 0$
Example: $u(t) = a_1 t^p + a_2 t^{2p} + ... + a_k t^{kp} + bt^{kp+1}$ is a map of degree p and its different has valuation k with any k possible.

Shafarevich theorems: III. Curves

– Curves of genus 1

(i) Elliptic curves over a number field finiteness: proven by Shafarevich using Siegel theorem on integral points.

(ii) Curves of genus 1 with no point and a fixed Jacobian E: A counterexample over $\mathbb{Q}$ has been given by Tate. But if you impose that that the curve has a rational point over the completion of K at every place one conjectures finiteness (Tate -Shafarevich, Birch and Swinnerton-Dyer) it is the famous Tate -Shafarevich group $Ш(E, K)$

Shafarevich theorems: III. Curves, cont.

– Curves of genus at least 2

(i) K a function field of characteristic zero: finiteness of $\text{Ш}(g, K, S)$ has been proved (Parshin and Arakelov)

(ii) K function field characteristic $p > 0$: Then $\text{Ш}(g, K, S, \text{ semi-stable})$ is finite (L.S)

(iii) But If you do not assume semi-stable one can find counterexamples to finiteness over infinite fields (L.S)

(iv) K number field: Then $\text{Ш}(g, K, S)$ is finite (Faltings). In fact the finiteness is proved for Abelian varieties of dimension g.

Shafarevich theorems: IV. Dynamical Systems

It is the subject of this talk. If one looks at the notion of bad reduction defined by Morton and Silverman one gets counterexamples to finiteness : The set of monic polynomials with coefficients in a ring of integers of a number field K has good reduction at every place.

Three notions of good reduction for self maps of $\mathbb{P}^1$: I. Definitions

For a morphism $\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ we note R_φ (respB_φ) the ramification locus (resp. the branch locus).

The choice of a v-lattice in a vector space of dimension 2 over K gives us a v-model $\mathbb{P}^1_{\mathcal{O}_v}$. When we have a v-model we say that a divisor D in $\mathbb{P}^1_K$ is étale at v if its schematic closure in $\mathbb{P}^1_{\mathcal{O}_K}$ is étale over $\operatorname{Spec} \mathcal{O}_v$

For example, if D is a reduced finite set of n points of $\mathbb{P}^1_K$, the schematic closure of their union is the corresponding set of n points of $\mathbb{P}^1_{\mathcal{O}_v}$ and it is étale at v if the reduction modulo the maximal ideal of v is made of distinct n points. This can also be said by:

$$|D| = |red_v D|$$

Three notions: II. Examples

- Example: A morphism defined by a monic polynomial has simple good reduction .
- Example: The Lattès map associated to an elliptic curve with Weierstrass equation $y^2 = P(x)$

$$\varphi(x) = \frac{P'^2 - 12xP}{4P}$$

has every type of good reduction at $v \neq 2, 3$ if v is a place of good reduction for the elliptic curve (i.e P still has 3 distinct roots mod v)

Three notions: II. Examples, cont.

-Example : Maps of degree 2 in normal form over the rational line:

$$\varphi = \frac{X^2 + \lambda XY}{\mu XY + Y^2}$$

with $\lambda = a + bt^N$ and $\mu = a^{-1} + b't^N$ and a finite number of conditions on a, b, b'.
Then the differential bad reduction is at least N+1 and the degree of the resultant is 2N. (Hope for an effective Shafarevich over function field)

Notes :
-A morphism $\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ extends as a morphism $\phi : \mathbb{P}^1_{\mathcal{O}_v} \to \mathbb{P}^1_{\mathcal{O}_v}$ if and only if φ has simple good reduction at v.
- If the valuation of the multiplier at a fix point is positive the ramification point mod v is also a branch point.

Three notions: IV. Comparisons

- Differential good reduction implies critical good reduction
- Critical good reduction is related to simple good reduction by the following:

Lemma (J.K.Canci, G. Peruginelli, D. Tossici)

Let $\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ a morphism of degree $\geqslant 2$ Let v be a non archimedean place of K. Suppose that the reduced map φ_v is separable. Then the following statements are equivalent:
(i) φ has critical good reduction at v:
(ii) φ has simple good reduction at v and $(B_\varphi)_{red}$ is étale at v.

Proof of the finiteness theorem

I. PGL_2 orbits of finite set of points with prescribed good reduction

Let $U \subset \mathbb{P}^1_K$ be a finite set. We shall say that U has good reduction outside of S if for $v \notin S$

$$|U| = |red_v(U)|.$$

It is equivalent to say that the schematic closure U_v of U in $\mathbb{P}^1_{\mathcal{O}_v}$ is étale over Spec $\mathcal{O}_v$ for any v $\notin S$.

We shall say that U is isotrivial if there exists a set $U' \subset \mathbb{P}^1_{\overline{k}}$ and $\gamma \in PGL_2(\overline{K})$ such that $\gamma(U) = U'$

Proof of the finiteness theorem

I. PGL_2 orbits of finite set of points with prescribed good reduction

Theorem

Fix a natural number N and a finite set of places S. Then $\text{Ш}(\mathbb{P}^1_K, N, S)$, the set of $\mathrm{PGL}(2)$ equivalence classes of non-isotrivial, $\mathrm{Gal}(\overline{K}/K)$-stable sets $U \subset \mathbb{P}^1_{\overline{K}}$ having good reduction outside of S and of cardinality equal to N, is finite.

Corollary

There exists a finite set $Y \subset \mathbb{P}^1_{\overline{K}}$ such that, for any U non-isotrivial, Gal($\overline{K}/K$)-stable , contained in $\mathbb{P}^1_{\overline{K}}$ having good reduction outside of S and cardinality equal to N , there exists an element $\gamma \in PGL_2(K)$ such that $\gamma(U) \subset Y$.

Proof of the finiteness theorem

II. Use of the S- unit theorem

Let $\mathcal{P}_{0,N}$ the functor that assign to any scheme X the set of distinct N-uples of X valued points on $\mathbb{P}^1$. This represented by the scheme :

$$(\mathbb{P}^1)^N - \textit{diagonals}$$

Let $\mathcal{M}_{0,N}$ be the quotient of $\mathcal{P}_{0,N}$ by the action of PGL(2). It is represented by the scheme:

$$(\mathcal{M}_{0,4})^{N-3} - \textit{diagonals}$$

Where

$$\mathcal{M}_{0,4} = \mathbb{P}^1 - \{0, 1, \infty\}$$

The moduli is given by the cross ratios:

$$(P_1, P_2, P_3, \ldots, P_N) \to ([P_1, P_2, P_3, P_i])_{i=4,5,\ldots,N}$$

Proof of the finiteness theorem

II. Use of the S-unit theorem

The cross ratios (which vanish if the points are not distinct) satisfy the following equation:

$$[A, B, C, D] + [A, C, B, D] = 1$$

Moreover the cross ratios are S-units because we consider only sets of N points having good reduction outside S. By the S-unit theorem the set $Ш(\mathbb{P}^1_K, N, S)$ is finite.

Proof of the finiteness theorem

III. Use Grothendieck computation of the tangent space to the scheme of morphisms

Theorem

Let X and Y be schemes of finite type over an algebraically closed field K and let Z be a closed subscheme of X defined by a sheaf of ideals $\mathfrak{I}_Z$. Fix a K morphism g: $Z \to Y$ and note $Hom_K(X, Y, g)$ the set of K morphism extending g. We have the following relation between tangent spaces for f a closed point of $Hom_L(X, Y, g)$:

$$T_{f, Hom_L(X,Y,g)} \cong H^0(X, f^* T_Y \otimes_{\mathcal{O}_X} \mathfrak{I}_Z)$$

Proof of the finiteness theorem

III. Use Grothendieck computation of the tangent space to the scheme of morphisms

Corollary

Let Y be a finite subset of $\mathbb{P}^1_K$ and let $d > 1$ be an integer. Suppose the characteristic of the field is 0 or is p and $p \geqslant d$. Then there are only finitely many morphisms $\varphi : \mathbb{P}^1_K \to \mathbb{P}^1_K$ of degree d satisfying all of the following conditions:
(i) $R_\varphi \subseteq Y$
(ii) $B_\varphi \subseteq Y$
(iii) $|R_\varphi| \geqslant 3$

References:

- L.S with T.Tucker A shafarevich Faltings Theorem for rational functions Pure and Applied Mathematics Quaterly Volume 4 Number 3
-L.S with Lloyd West A Dynamical Shafarevich Theorem for ratinal maps over Number Fields and Functions Fields ArXiv 16th May 2017 (1705 05 48 9v 1)