Optimal subvarieties and raising to the power i

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Plan

1. ZP; connection with SC
2. Optimal subvarieties: a reformulation of ZP
3. Uniform ZP
4. Raising to the power i

Bottom line: an analogue of ZP for $w = z^i$ can be proved, because all arithmetic difficulties disappear thanks to the Gelfond-Schneider theorem.

1. The Zilber-Pink Conjecture

A conjecture with 3 sources:

Zilber: model theory of exponentiation
Pink (most general form): unifying ML, AO, André
Bombieri-Masser-Zannier: exploring problems from Schinzel

Variety: irreducible (relatively) closed algebraic set defined over $\mathbb{C}$.

ZP involves:

Ambient variety X e.g. $\mathbb{G}_{\mathrm{m}}^n$, $Y(1)^n$, Shimura variety, MSV;
Its collection $\mathcal{S}$ of "special subvarieties" ;
A subvariety $V \subset X$;

ZP (conjecturally) governs intersections $V \cap T$ with $T \in \mathcal{S}$.

Special subvarieties

Multiplicative setting $X = \mathbb{G}_{\mathrm{m}}^n = (\mathbb{C}^\times)^n$

Special subvarieties are the (irreducible) subvarieties defined by **multiplicative relations**, also known as **torsion cosets**:
Impose finitely many relations

$$X_1^{a_{1j}} \ldots X_n^{a_{nj}} = 1, a_{ij} \in \mathbb{Z},\ \mathrm{naz},\ j \in J,$$

and take components. Special points=torsion points $(\zeta_1, \ldots, \zeta_n)$.

Modular setting $X = Y(1)^n = \mathbb{C}^n$

Special subvarieties are the (irreducible) subvarieties defined by **modular relations**:
Impose finitely many relations

$$\Phi_{N_{ij}}(X_i, X_j) = 0, (i,j) \in L$$

(or and finitely many $X_k = \sigma_k, k \in L$ with singular moduli σ_k), and take components. Special pts $= (\sigma_1, \ldots, \sigma_n)$, σ_i singular moduli.

Atypical subvarieties

Fix $V \subset X$. Let $T \in \mathcal{S}$. Let $A \subset_{\text{cpt}} V \cap T$. Expect:

$$\dim A = \dim V + \dim T - \dim X$$

(its never less). If $\dim A$ is bigger, call A **atypical** for V.
(aka: "anomalous", or "unlikely" if expect $V \cap T = \emptyset$)

Definition. The **atypical set** of V is the union of all atypical sbvs.

A priori the atypical set is a countable union of subvarieties.

ZP Conjecture. The atypical set is a **finite** union.

Pink: most general form, for a mixed Shimura variety X and its collection $\mathcal{S}$ of special subvarieties (though only for "unlikely" intersections).

Schanuel's conjecture

Zilber's motivation for ZP.

Schanuel's conjecture: For $z_1, \ldots, z_n \in \mathbb{C}$,

$$\mathrm{tr.deg.}_{\mathbb{Q}}\mathbb{Q}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \geq \mathrm{l.d.}_{\mathbb{Q}}(z_1, \ldots, z_n).$$

Reformulation: For every $V \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$, $V/\overline{\mathbb{Q}}$, $\dim V < n$, if $(z, e^z) \in V$ then $z \in L$ for some proper rational subspace $L \subset \mathbb{C}^n$.

For a given V, can we hope for some finiteness concerning $\{L\}$?

Easy examples show we cannot get: a finite collection of L, for every given V.
But the "exceptional" case in SC does lead to an atypical intersection.

Schanuel's conjecture and atypical intersections

Let V as before: $V \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n, V/\overline{\mathbb{Q}}, \dim V < n$. **Assume SC.**

Suppose $(z, e^z) \in V$, with $e^z \in W = \pi_{\mathrm{m}}(V)$ such that the fibre over e^z is of generic dimension $\dim V - \dim W$.

Say $z \in L$, $\mathbb{Q}$-subspace, with $\dim L = \dim_{\mathbb{Q}}(z)$ and $T = \exp L$ "special" (torus), and $e^z \in A \subset_{\mathrm{cpt}} W \cap T$. Then:

$$\dim L = \dim T \leq_{\mathbf{SC}} \mathrm{tr.d.}(z, e^z) \leq \dim A + (\dim V - \dim W),$$

which implies (as $\dim V < n$):

$$\dim A \geq \dim T + \dim W - \dim V > \dim T + \dim W - n$$

and so e^z lies in an atypical intersection $A \subset W \cap T$

Uniform SC

USC. Let $V \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$, $V/\overline{\mathbb{Q}}$, $\dim V < n$. There is a finite set $\{L\}$ of $\mathbb{Q}$-subspaces $L \subset \mathbb{C}^n$ and finite set $\{T\}$ of special $T \subset (\mathbb{C}^\times)^n$ of codimension at least 2 such that if $(z, e^z) \in V$ then either $z \in L$ for some $L \in \{L\}$ or $e^z \in T$ for some $T \in \{T\}$.

Remarks:

1. SC+ZP implies USC (previous page)
2. USC implies SC (because of the "at least 2")
3. USC seems not to imply ZP, in which the set of $\{T\}$ is independent of V.

2. Optimal subvarieties

Let $A \subset X$ with special subvarieties $\mathcal{S}$.

Then there is smallest special subvariety $\langle A \rangle$ containing A, and define the **defect** (after Pink)

$$\delta(A) = \dim\langle A \rangle - \dim A.$$

Fix $V \subset X$. Call $A \subset V$ **optimal** (for V) if it is maximal for its defect among subvarieties of V. I.e. if $A \subset B$ proper with $B \subset V$ then $\delta(B) > \delta(A)$.

ZP (optimal formulation): Let $V \subset X$. Then V has only finitely many optimal subvarieties. (V is one).

Optimal subvarieties: remarks

Introduced in paper with Habegger proving analogue of Maurin's theorem for curves in abelian varieties, and giving a conditional proof of full ZP in modular and abelian setting.

Also corresponding notion "geodesic optimal" w.r.t. weakly special subvarieties, appeared in earlier model-theoretic work of Poizat as "cd-maximal".

A nice notion as it is intrinsic to V, while being atypical can depend on whether V is contained in a proper special or not.

Maurin's Theorem. *Suppose $V \subset \mathbb{G}_{\mathrm{m}}^n$ a curve, $V/\overline{\mathbb{Q}}$ and not contained in a proper special subvariety. Then $V \cap \bigcup T$ is finite, the union over special subvarieties of codimension at least 2.*

BMZ: Before: V not in proper weakly special; and after: $V/\mathbb{C}$.

Optimal points

$+$ Habegger showed: ZP for $\mathbb{G}_{\mathrm{m}}^n, Y(1)^n$ (and A) reduces to finiteness of optimal points (for all $V \subset \mathbb{G}_{\mathrm{m}}^k, k \leq n$, respectively $V \subset Y(1)^k, k \leq n$).

These points are then algebraic over a field of definition for V, and the required hypthesis for ZP is a Galois orbit lower bound.

Then: o-minimality, point-counting, and (modular) Ax-Schanuel (+Tsimerman, 2016; abelian Ax-Schanuel is a theorem of Ax).

Daw and Ren, 2017: generalize this to general Shimura case of ZP, reduce it to: Ax-Schanuel for Shimura vars (Mok+P+Tsimerman, 2017), and arithmetic conjectures (so far not only Galois lower bounds), via definability (Peterzil-Starchenko, Klingler-Ullmo-Yafaev), o-minimality and point-counting.

3. Uniformity in ZP

Scanlon (*IMRN*) showed that AO (and ML) is "automatically" uniform over families of algebraic varieties $V_t \subset X, V \subset X \times P$.

One way to express this is that the "special set" (union of special subvarieties) is bounded as a cycle over such V_t.

Another (Scanlon): Exists another family $W_s \subset X, W \subset X \times Q$ with: $\forall T \exists s : \mathrm{Opt}(V_t) = W_s$.

UZP: Let $V \subset X \times P$ be a family of algebraic subvarieties V_t, parameterized by $t \in P$. Then the "optimal cycle" $\mathrm{Opt}(V_t)$ is bounded uniformly for $t \in P$.

Sketch by Zannier (*Annals Studies*): uniformity for curve $V \subset \mathbb{G}_{\mathrm{m}}^n$, not contained in a proper weakly special (theorem of BMZ).

Masser (*ibid*): Uniformity for lines in $\mathbb{G}_{\mathrm{m}}^3$.

Stoll (*JEMS*, t.a.): Special cases of uniformity in ML (question of Mazur), implications and unconditional results.

ZP implies UZP

Theorem. *For $Y(1)^n$ and $\mathbb{G}_{\mathrm{m}}^n$, ZP implies UZP.*

Sketch. Using reduction to finiteness of optimal points, it suffices to show that in a family of varieties $V \subset X \times P$ with fibres $V_t \subset X$, the number of optimal points is uniformly bounded.

Show that (following Zannier) for large N, a V_t with N optimal points leads to an atypical point on the "incidence variety"

$$W = \{\bar{z}_1, \ldots, \bar{z}_N \in X^N : \exists t : \bar{z}_i \in V, i = 1, \ldots, N\}.$$

Apply ZP in X^N. Leads to an induction over families of V in families of weakly special subvarieties of X, via combinatorial principles. □

Note. This also shows (known by Zilber, and by BMZ in a very precise form) that ZP for $V/\overline{\mathbb{Q}}$ implies ZP for $V/\mathbb{C}$ (implies UZP).

4. Raising to the power i

The multi-valued function

$$(z, w) \in \Gamma \iff w = z^i \iff \exists u : e^u = z \wedge e^{iu} = w.$$

Model theory of raising to powers: studied by Zilber, formulated the corresponding "SC" which we will also formulate (for z^i), though a bit differently.

Recently: quasiminimality of $(\mathbb{C}, +, \times, \Gamma)$ proved by Wilkie (this structure is not o-minimal).

Quasiminimal: definable subsets of $\mathbb{C}$: countable or co-countable.

Quasiminimality of $(\mathbb{C}, +, \times, exp)$ is an open conjecture of Zilber; it is unknown even whether $\mathbb{R}$ is definable there.

Towards SC for z^i

Say $(z_1, w_1), \ldots, (z_n, w_n) \in \Gamma$ with logs $u_1, \ldots, u_n$ (unique).
Then SC asserts:

$$\mathrm{tr.deg.}_{\mathbb{Q}}(u_1, \ldots, u_n, iu_1, \ldots, iu_n, z_1, \ldots, z_n, w_1, \ldots, w_n) \geq 2n$$

unless $u_1, \ldots, u_n, iu_1, \ldots, iu_n$ are l. dep $/\mathbb{Q}$.
And therefore

$$\mathrm{tr.deg.}_{\mathbb{Q}}(z_1, \ldots, z_n, w_1, \ldots, w_n) \geq n$$

unless $z_1, \ldots, z_n, w_1, \ldots, w_n$ are mult. dep.
However

1. $\overline{z}, \overline{w}$ might be mult. dep. when u_j, iu_j are not l dep $/\mathbb{Q}$
2. If u_j, iu_j are l dep $/\mathbb{Q}$ there is then a second linear relation:

$$\sum q_j u_j + \sum r_j iu_j = 0 \rightarrow -\sum r_j u_j + \sum q_j iu_j = 0 \quad (\text{and} \ \leftarrow)$$

SC for z^i

Definition. A **plu-torus** $T \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ is a torus whose lattice of defining exponent vectors $L \subset \mathbb{Z}^{2n}$ is closed under $(\bar{a}, \bar{b}) \to (-\bar{b}, \bar{a})$.

SC for z^i: Let $(x_i, y_i) \in \Gamma$, $i = 1, \dots, n$. Then

$$\mathrm{tr.deg.}_{\mathbb{Q}}\mathbb{Q}(\bar{x}, \bar{y}) \geq \frac{1}{2} \dim((\bar{x}, \bar{y}))_{\mathrm{plu}}.$$

Then: SC implies SC for z^i.

USC for z^i. Let $T \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ be a plu-torus. Let $V \subset T$ with $\dim V < n$, $V/\overline{\mathbb{Q}}$. There is a finite set $\mathcal{U} = \mathcal{U}(V)$ of proper plu-sub-tori $U \subset T$ such that if $(x, y) \in V \cap \Gamma$ then $(x, y) \in U$ for some $U \in \mathcal{U}$.

USC for z^i

Ideologically would like: a statement S with

$$z^i\mathrm{SC} + \mathrm{S} \quad \to z^i\mathrm{USC}.$$

We will formulate and prove a statement z^iZP with:

$$\mathrm{SC} + z^i\mathrm{ZP} \quad \to z^i\mathrm{USC}.$$

The collection of plu-subtori

We want to consider the collection of plu-tori as "special subvarieties" of $\mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$.

The intersection of two tori is not in general a torus, but has a unique torus component. So we get a suitable "special collection" if we restrict to varieties and components intersecting Γ^n.

I.e. If $V \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ with $V \cap \Gamma^n \neq \emptyset$ then there is a smallest plu-subtorus $((V))_{\mathrm{plu}}$ containing V. This is since if $(x, y) \in V$ with $(x, y) \in T_1 \cap T_2$ then its unique logarithm lies in the corresponding linear spaces $L_1, L_2/\mathbb{Q} \subset \mathbb{C}^n$.

So we define the **plu-defect** $\delta_{\mathrm{plu}}(A) = \dim((A))_{\mathrm{plu}} - \dim A$, for $A \subset \mathbb{G}_{\mathrm{m}}^n \times \mathbb{G}_{\mathrm{m}}^n$ meeting Γ^n.

And for V we define **plu-optimal** $A \subset V$.

ZP for z^i

Theorem. *Let $V \subset \mathbb{G}_m^n \times \mathbb{G}_m^n$. Then V contains only finitely many plu-optimal subvarieties. And this is uniform in families.*

Sketch idea of proof. The main point is this. Say $V/\overline{\mathbb{Q}}$.

The typical "atypical intersection" is a **point** of "unlikely intersection", hence an **algebraic point** on Γ.

But an algebraic point $(z, w) \in \Gamma$ is just $(1, 1)$ by the Gelfond-Schneider theorem.

So we reduce the general case for $V/\overline{\mathbb{Q}}$ to optimal points, and then prove the result is uniform in families for families $V/\overline{\mathbb{Q}}$, which gives the general case. □

THANK YOU!

And finally:

Happy Birthday Umberto,
with all best wishes.