Finite Monodromy in Finite Characteristic

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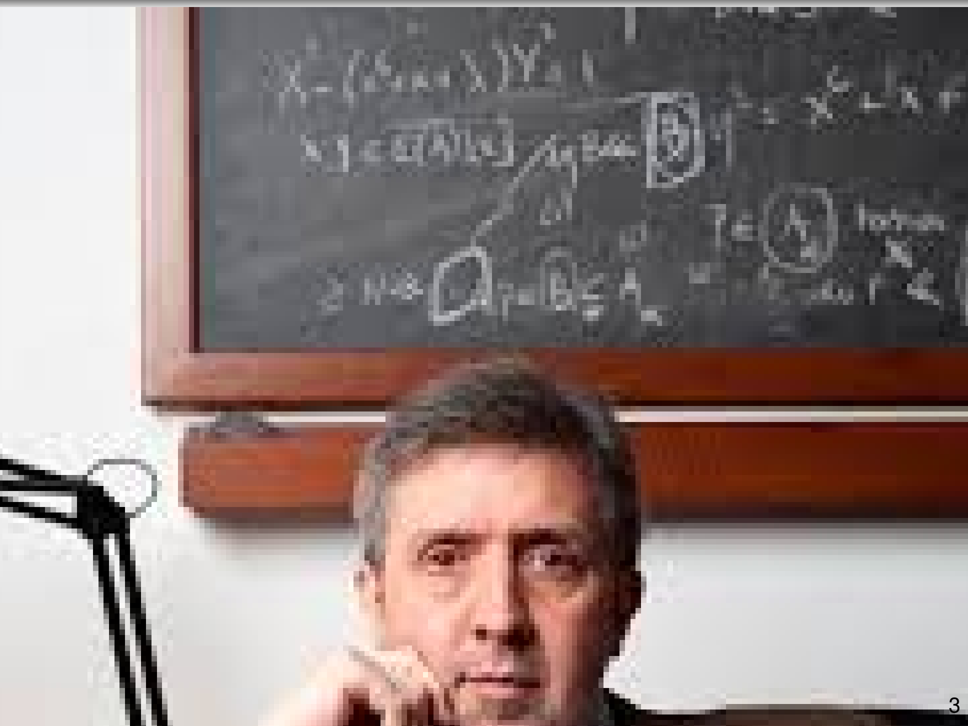
I begin with picture of the birthday boy.



The title of the conference is

Specialization Problems in Diophantine Geometry.

In this talk, it is the characteristic we will be specializing.

To make the connection to my title, consider the one-parameter (t the parameter) family of elliptic curves

$$y^2 - y = x^3 + tx.$$

If we look at this family in any characteristic other than 2 or 3, we see a non constant *j*-invariant, and hence an ℓ-adic ($\ell \neq p$) monodromy group which is open in $SL(2, \mathbb{Z}_\ell)$.

However, in these two characteristics 2 and 3, this family has finite monodromy, because all members are supersingular: in characteristic 2, the Hasse invariant is the coefficient of xy in the equation, and in characteristic 3 it is the coefficient of x^2.

For this example in characteristic 2, Artin-Schreier theory tells us that for ψ the unique nontrivial additive character of $\mathbb{F}_2$, extended to finite extensions $k/\mathbb{F}_2$ by composition with the trace, we have

For $t \in k$, the trace of $Frob_k$ on the H^1 of the curve$/k$ given by $y^2 - y = x^3 + tx$ is the character sum

$$-\sum_{x \in k} \psi(x^3 + tx).$$

This is an instance of what is arguably the simplest sort of local system in characteristic $p > 0$.

Take a finite field k of characteristic p, a nontrivial additive character ψ of k, an integer $D \geq 3$ which is prime to p, and look at the character sums, one for each $t \in k$,

$$t \in k \mapsto \quad -\sum_{x \in k} \psi(x^D + tx).$$

with a similar recipe over finite extensions.

We can also "decorate" these sums by choosing a multiplicative character χ of $k^\times$ and looking at the sums

$$t \in k \mapsto \ -\sum_{x \in k} \chi(x)\psi(x^D + tx).$$

[The convention here is that $\mathbb{1}(0) = 1$ but $\chi(0) = 0$ for χ nontrivial.]

These sums are the trace function of a lisse $\overline{\mathbb{Q}_\ell}$-sheaf on $\mathbb{A}^1/k$ (any $\ell \neq p$)

$$\mathcal{F}(k, \psi, \chi, D).$$

It is pure of weight one, and has

$$\text{rank } = D - 1 \text{ for } \chi = \mathbb{1},$$

$$\text{rank } = D \text{ for } \chi \neq \mathbb{1}.$$

Its determinant $\det(\mathcal{F})$ is geometrically trivial (this uses $D \geq 3$).

This local system is geometrically irreducible and rigid because it is the Fourier transform of the rank one object $\mathcal{L}_{\chi(x)} \otimes \mathcal{L}_{\psi(x^D)}$, and Fourier transform preserves both these properties.

One knows that when the characteristic p is large compared to D, then the geometric monodromy group of this $\mathcal{F}$ is a connected, semisimple algebraic group over $\overline{\mathbb{Q}_\ell}$, either SO or SL or Sp, with the extra possibility of G_2 when $D = 7$. For example

$$\mathcal{F}(k, \psi, \mathbb{1}, \text{odd } D) \ : Sp(D-1)$$

$$\mathcal{F}(k, \psi, \mathbb{1}, \text{even } D) \ : SL(D-1)$$

$$\mathcal{F}(k, \psi, \chi_2, \text{even } D) \ : SL(D)$$

$$\mathcal{F}(k, \psi, \chi_2, \text{odd } D \ \neq 7) \ : SO(D)$$

$$\mathcal{F}(k, \psi, \chi_2, 7) \ : G_2$$

when $p >> D \geq 3$.

Back in 1986, Dan Kubert was coming to my graduate course, and in it he explained a method of proving that certain of the $\mathcal{F}(k, \psi, \chi, D)$ had finite geometric monodromy groups. They include

$$\mathcal{F}(\mathbb{F}_q, \psi, \mathbb{1}, q+1),$$

$$\mathcal{F}(\mathbb{F}_q, \psi, \mathbb{1}, (q+1)/2),\ q \text{ odd}$$

$$\mathcal{F}(\mathbb{F}_q, \psi, \chi_2, (q+1)/2),\ q \text{ odd}$$

$$\mathcal{F}(\mathbb{F}_q, \psi, \mathbb{1}, (q^n+1)/(q+1)),\ n \text{ odd},$$

$$\mathcal{F}(\mathbb{F}_{q^2}, \psi, \chi, (q^n+1)/(q+1)),\ n \text{ odd},\ \chi \neq \mathbb{1},\ \chi^{q+1} = \mathbb{1}.$$

In hindsight, I had already seen some of these, but only very recently did I understand that those that I had seen fell under Kubert's results. They were

$$\mathcal{F}(\mathbb{F}_2, \psi, \mathbb{1}, 3),$$

a $q + 1$ case, the elliptic curve family we started off with,

$$\mathcal{F}(\mathbb{F}_5, \psi, \chi_2, 3),$$

a $(q + 1)/2$ case, which gave $PSL(2, 5)$, but which I had "seen" as A_5,

$$\mathcal{F}(\mathbb{F}_3, \psi, \chi_2, 7),$$

a $(q^3 + 1)/(q + 1)$ case, which gave $SU(3, 3)$, a finite subgroup of G_2, and

$$\mathcal{F}(\mathbb{F}_{13}, \psi, \chi_2, 7),$$

a $(q + 1)/2$ case, which gave $PSL(2, 13)$, another finite subgroup of G_2.

For$q$ odd, the local system

$$\mathcal{F}(\mathbb{F}_q, \psi, \mathbb{1}, (q+1)/2)$$

has rank

$$(q-1)/2,$$

and the local system

$$\mathcal{F}(\mathbb{F}_q, \psi, \chi_2, (q+1)/2)$$

has rank

$$(q+1)/2.$$

For $q \geq 5$ odd, the group $SL(2, q)$ has, after the trivial representation, two irreducible representations of dimension

$$(q-1)/2,$$

and it has two of dimension

$$(q+1)/2.$$

For n odd, the local system

$$\mathcal{F}(\mathbb{F}_q, \psi, \mathbb{1}, (q^n+1)/(q+1)),\ n \text{ odd},$$

has rank

$$(q^n+1)/(q+1)-1,$$

and each of the q local systems

$$\mathcal{F}(\mathbb{F}_{q^2}, \psi, \chi, (q^n+1)/(q+1)),\ n \text{ odd},\ \chi \neq \mathbb{1},\ \chi^{q+1} = \mathbb{1},$$

has rank

$$(q^n+1)/(q+1).$$

For n odd and with the exception of $(n=3, q=2)$, the group $SU(n,q)$ has, after the trivial representation, one irreducible representation of dimension

$$(q^n+1)/(q+1)-1,$$

and it has q irreducible representations of dimension

$$(q^n+1)/(q+1).$$

THIS CANNOT BE AN ACCIDENT.

We formulate the obvious conjecture: that the geometric monodromy group is what the numerology suggests:

The geometric monodromy group for

$$\mathcal{F}(\mathbb{F}_q, \psi, \mathbb{1}, (q+1)/2),$$

the image of $SL(2,q)$ in one of its irreducible representations of dimension $(q-1)/2$; and for

$$\mathcal{F}(\mathbb{F}_q, \psi, \chi_2, (q+1)/2)$$

the geometric monodromy group is the image of $SL(2,q)$ in one of its irreducible representations of dimension $(q+1)/2$;

[And in both cases you get the other representation of the same dimension by changing ψ to $x \mapsto \psi(ax)$ for $a \in \mathbb{F}_q^\times$ a nonsquare.]

For n odd and with the exception of ($n = 3, q = 2$), the geometric monodromy group of

$$\mathcal{F}(\mathbb{F}_q, \psi, \mathbb{1}, (q^n + 1)/(q + 1)), \ n \,\mathrm{odd},$$

is the image of $SU(n, q)$ in its unique irreducible representation of dimension $(q^n + 1)/(q + 1) - 1$;

and for each of the q local systems

$$\mathcal{F}(\mathbb{F}_{q^2}, \psi, \chi, (q^n + 1)/(q + 1)), \ n \,\mathrm{odd}, \ \chi \neq \mathbb{1}, \ \chi^{q+1} = \mathbb{1},$$

it is the image of $SU(n, q)$ in one of its q irreducible representations of dimension $(q^n + 1)/(q + 1)$;

[And, when q is odd, taking $\chi = \chi_2$ should give the unique irreducible representations of dimension $(q^n + 1)/(q + 1)$ which is orthogonal.]

Here is the current status of these conjectures.

In the $SL(2,q)$ case, it is known for $q = p \geq 5$, using group theory results of Brauer, Feit, and Tuan that go back fifty years. [The only geometric inputs are that the geometric monodromy representation is unimodular, primitive (not induced), of dimension $(p \pm 1)/2$, and the image group has order divisible by p.]

The situation for $SL(2, q)$, $q \geq 5$ odd, is more complicated, and relies heavily on work of Dick Gross, itself based on Deligne-Lusztig. This work gives a good handle on the representations which factor through $PSL(2, q)$, which are those of ours whose dimension $(q \pm 1)/2$ is **odd**.
There is then a trick to pass to the other ones of ours, those whose dimension $(q \pm 1)/2$ is even.

Here is the trick. Of the two local systems, the “small one" has dimension one less than the big one.

The fact is that Sym^2 of the small one is (isomorphic to) Exterior^2 of the (correctly chosen) big one.

This statement amounts to a list of exponential sum identities. I was able to prove them when 2 was a square in $\mathbb{F}_q$, but not otherwise. So I consulted the master of exponential sum identities, Ron Evans, who did the other case.

The situation for $SU(n, q)$, $n \geq 3$ odd, is this.

For $n \geq 5$ odd, nothing is known.

For $n = 3$ and $q \geq 3$, we know nothing when q is even. When q is odd, what we do know is again based on (the same) work of Dick Gross. This gives us a good handle on those representations that factor through $PSU(3, q)$.

Fortunately, the group $SU(3,q)$ has a trivial center unless q is 2 mod 3. Thus when q is not 2 mod 3, the groups $SU(3,q)$ and its quotient $PSU(3,q)$ coincide, and the conjecture is known for $SU(3,q)$. When q is 2 mod 3, then we know the conjecture for

$$\mathcal{F}(\mathbb{F}_q, \psi, \chi_2, (q+1)/2)$$

and for those

$$\mathcal{F}(\mathbb{F}_{q^2}, \psi, \chi, (q^n+1)/(q+1)),\ n \text{ odd},\ \chi \neq \mathbb{1},\ \chi^{q+1} = \mathbb{1},$$

whose χ has $\chi^{(q+1)/3} = \mathbb{1}$.

Here is a case where we do NOT KNOW the monodromy is finite, but computer experiments suggest that it is:

$$D = 2q - 1, \chi = \chi_2,$$

the quadratic character. So the sums we are looking at are

$$t \in k \mapsto \ -\sum_{x \in k} \chi_2(x)\psi(x^{2q-1} + tx).$$

The only case of these that comes under the umbrella of what we know is the case $q = 3$. Thus $D = 2q - 1 = 5$. This is the $(Q + 1)/2$ case for $Q = 9$, where we have proven the monodromy group to be $PSL(2, 9)$.

Meanwhile, Guralnick and Tiep tell me that IF the monodromy is finite, then the monodromy group is the alternating group $Alt(2q)$, in its “deleted permutation" representation of dimension $2q - 1$.

How does this square up with what we have in the $q = 3$ case, where the group is known to be $PSL(2, 9)$?
All is well, because $Alt(2q = 6)$ is isomorphic to $PSL(2, 9)$.

MUCH REMAINS TO BE DONE.