On the Betti map associated with abelian logarithms

Pietro Corvaja - Università di Udine

(after a joint work with Yves André and Umberto Zannier)

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These coordinates are called *Betti coordinates*.
We denote them by $(\beta_1, \ldots, \beta_{2g}) \in \mathbb{R}^{2g}$.

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$$\int_0^{\xi} \omega_j = \sum_{i=1}^{2g} \beta_i \int_{\gamma_i} \omega_j, \qquad j = 1, \ldots, g\,.$$

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The rational values of β correspond to torsion values of ξ.

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If ξ is not identically torsion, for every non-empty open set $U \subset S(\mathbb{C})$ there exists an integer n_0 such for all $n > n_0$ there exists a point $s \in U$ such that $\xi(s)$ is a torsion point of order n.

An application: the elliptical billiard

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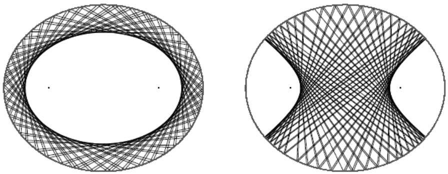

All trajectories are tangent to another conic, called the *caustic*.

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Changing the direction of the first shot determines a variation of the elliptic curve E, so an algebraic family of elliptic curves provided with a section.

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This last fact can be proved by considering an n-gon inscribed in the ellipse of maximal length. In turn, one reproves in this way the non-constancy of the Betti map (Manin's theorem).

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We prove that under some 'generically satisfied' conditions *on the family* $\mathcal{A} \to S$, not involving the section, the Betti map of every section not contained in a proper subgroup scheme has maximal rank.

We fix a basis $(\omega_1, \ldots, \omega_g)$ of global sections of $\Omega^1_{\mathcal{A}}$, and complete it into a symplectic basis $(\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$ of $\mathcal{H}^1_{dR}(\mathcal{A}/S)$

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We set $\Omega_1 := \left(\int_{\gamma_i} \omega_j\right)_{i,j=1,\dots g}$, $\Omega_2 := \left(\int_{\gamma_{i+g}} \omega_j\right)_{i,j=1,\dots g}$,

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The $g \times g$ matrix Z takes values in the Siegel's space $\mathcal{H}_g$, and is a holomorphic map $\tilde{S} \to \mathcal{H}_g$.

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Theorem *Assume* $\dim S = g$*, the family has no constant part and the section is not contained in any proper subgroup scheme. If the Betti map of the section is not a submersion, then for every vector* $\mu \in \mathbb{C}^g$ *and every* $\tilde{s} \in \tilde{S}$*, there exists a non-zero derivation* $\partial \in T_{\tilde{S}}(\tilde{s})$ *such that* $\partial(Z \cdot \mu) = 0$*.*

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This result can be interpreted in the frame of the Kodaira Spencer map.

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Theorem *Let us suppose that $g \leq 3$. Assume the monodromy of $\mathcal{A} \to S$ is Zariski-dense in Sp_{2g} and that ξ is not contained in a proper subgroup scheme of $\mathcal{A}$. Then*

$$\mathrm{rk}\ \beta = 2 \min(d_{\mu_A}, g), \tag{0.1}$$

where d_{μ_A} is the dimension of the image of modular map $\mu_A : S \to \mathcal{A}_g$.

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Theorem *Let S be a finite cover of $(\mathbb{P}^1 \setminus \{0,1,\infty\})^{2g-1}$, $\xi : S \to \mathcal{A}$ be any non-torsion section. After replacing S by a suitable dense Zariski-open subset, β becomes a submersion.*

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Theorem *The set of real points $s = (s_1, \ldots, s_{2g}) \in \mathbb{R}^{2g}$ such that $\xi(s)$ is torsion is dense in the euclidean topology.*

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Identifying $\mathrm{Lie}\,A$ and $\mathrm{Lie}\,A^\vee$ via polarization, we obtain that $\theta_{A,\partial}$ is a symmetric endomorphism of $(\mathrm{Lie}\,A)^\vee$.

Description in terms of the period maps

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$$Y := \begin{pmatrix} \Omega_1 & N_1 \\ \Omega_2 & N_2 \end{pmatrix}$$

where $\Omega_1, \Omega_2, N_1, N_2$ are the $(g \times g)$-matrices

$$\Omega_1 = \left(\int_{\gamma_i} \omega_j \right)_{i,j}, \Omega_2 = \left(\int_{\gamma_{i+g}} \omega_j \right)_{i,j}, N_1 = \left(\int_{\gamma_i} \eta_j \right)_{i,j}, N_2 = \left(\int_{\gamma_{i+g}} \eta_j \right)_{i,j}$$

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Here $(\omega_1, \ldots, \omega_g, \eta_1, \ldots, \eta_g)$ is a symplectic basis of $\mathcal{H}^1_{dR}(\mathcal{A}/S)$.

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Differential equation for Y: for every derivation ∂ in S

$$\partial Y = Y \cdot \begin{pmatrix} R_\partial & S_\partial \\ T_\partial & U_\partial \end{pmatrix}$$

where $R_\partial, S_\partial, T_\partial, U_\partial$ are matrices with $\mathcal{O}_S(S)$.

Viewing the Kodaira-Spencer map as a morphism

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the symmetric matrix T_∂ is the matrix of the Kodaira Spencer map with respect to the basis $(\omega_1, \ldots, \omega_g)$ of $(\operatorname{Lie} \mathcal{A})^\vee$ and $\eta_1, \ldots, \eta_g$ of $\operatorname{Lie} \mathcal{A}$.

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In the case $\dim S = g$, the condition in our result can be rephrased:

Viewing the Kodaira-Spencer map as a morphism

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the symmetric matrix T_∂ is the matrix of the Kodaira Spencer map with respect to the basis $(\omega_1, \dots, \omega_g)$ of $(\mathrm{Lie}\,\mathcal{A})^\vee$ and $\eta_1, \dots, \eta_g$ of $\mathrm{Lie}\,\mathcal{A}$.

In the case $\dim S = g$, the condition in our result can be rephrased:

If the Betti map does not have generically maximal rank $2g$, then

$$\forall \omega \quad \exists \partial \neq 0, \quad \theta_\partial(\omega) = 0.$$