Galois theory of periods, and the André-Oort conjecture

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Outline of Galois theory of periods

$\int_\Delta \omega$, Δ and ω "algebraic"

(ω diff. form on an algebraic variety X defined over some number field k, $\Delta \subset X(\mathbb{R})$ defined by algebraic inequations $/k$),

Transcendence of periods? Algebraic relations between them (period relations)?

Leibniz (1691, letters to Huygens): speculation about transcendence of π and some other (1-dim) periods. Inquiry about "accidental" cases when they are algebraic: "nothing happens without a reason"...

General conjectures:

Grothendieck ('66): *any period relation is of motivic origin.*

X smooth $/k$, $H_*(X(\mathbb{C}), \mathbb{Q}) \otimes H^*_{dR}(X) \to \mathbb{C}$

expressed by period matrix Ω_X.

If X proper, Z alg. subvariety dim. r of X^m ,

$\omega \in H^{2r}_{dR}(X^m) \subset H_{dR}(X)^{\otimes m} \rightsquigarrow \int_Z \omega \in (2\pi i)^r k$

conjecturally, period relations always come in this way.

Kontsevich ('98) (-Zagier): *any period relation comes from the basic rules for* $\int$:

linearity, product, algebraic change of variable $\int_{\Delta} f^*\omega = \int_{f_*\Delta} \omega$, Stokes $\int_{\Delta} d\omega = \int_{\partial\Delta} \omega$.

When made precise, these two conjectures can be proven to be equivalent.

Remark. Functional analog of periods:

$$\mathbb{Q} \rightsquigarrow \mathbb{C}(t).$$

Ayoub (2015) *proved* analogs of Grothendieck's and Kontsevich's conjectures in this case.

Motives: categorification of the Grothendieck ring of varieties $K_0(Var_k)$.

$\rightsquigarrow$ abelian $\otimes$-category $MM(k)$.

3 unconditional, compatible theories:

A. (pure motives, ie. motives attached to projective smooth varieties), Nori, Ayoub; *cf.* Bourbaki nov. 2015).

eg. $X \rightsquigarrow$ motive of $X \rightsquigarrow \langle X \rangle_{\otimes} \cong Rep_{\mathbb{Q}} G_X$

$G_X \subset GL(H(X(\mathbb{C})), \mathbb{Q})$ motivic Galois group of X.

Example. X = abelian variety $/\mathbb{C}$.

$$H^1(X(\mathbb{C})),\mathbb{Q})\otimes\mathbb{C} = \Omega^1(X)\oplus\overline{\Omega^1(X)}$$

$$c_{\lambda\mu} := \lambda\cdot id_{\Omega^1(X)} + \mu\cdot id_{\overline{\Omega^1(X)}}.$$

Fact (A. '96): G_X is isomorphic to the Mumford-Tate group of X, ie. the smallest algebraic $\mathbb{Q}$-subgroup

$$H\subset GL(H^1(X(\mathbb{C})),\mathbb{Q})$$

such that $\forall\lambda,\mu\in\mathbb{C}^\times$, $c_{\lambda\mu}\in H(\mathbb{C})$.

eg. X = non CM elliptic curve: $G_X = GL_2$,
X = CM elliptic curve by K: $G_X(\mathbb{Q}) = K^\times$.

$\langle X\rangle_{\otimes} \overset{H_B, H_{DR}}{\rightarrow} Vec_{\mathbb{Q}}$ $(k = \mathbb{Q}) \rightsquigarrow \Pi_X$ period torsor

Period pairing $\Leftrightarrow$ canonical point in $\Pi_X(\mathbb{C})$:

$$\mathrm{Spec}\,\mathbb{C} \overset{\varpi_X}{\rightarrow} \Pi_X.$$

Grothendieck's period conjecture:

PC$_X$: ϖ_X *is a generic point.*

Equivalently: Π_X *is connected, and*

$$\mathrm{TrDeg}_{\mathbb{Q}}\,\mathbb{Q}[\Omega_X] = \dim G_X.$$

If so, one can develop a bit of Galois theory of periods: $G_X(\mathbb{Q})$ would act on $\mathbb{Q}[\Omega_X] \rightsquigarrow$ *Conjugates* of periods...

Examples: $X = \mathbb{P}^1$: $G_X(\mathbb{Q}) = \mathbb{Q}^{\times}$,
$\mathbb{Q}[\Omega_X] = \mathbb{Q}[2\pi i]$ (**PC**$_X$: Lindemann),

X = CM elliptic curve by K: $G_X(\mathbb{Q}) = K^{\times}$,
$\mathbb{Q}[\Omega_X] = \mathbb{Q}[\omega_1, \eta_1]$ (**PC**$_X$: Chudnovsky).

What if $k \subset \mathbb{C}$ is no longer algebraic over $\mathbb{Q}$?

generalized $\mathbf{PC}_X$: $\mathrm{TrDeg}_{\mathbb{Q}}\, k[\Omega_X] \geqslant \dim G_X$

(A. '97).

Should hold for any motive; in the case of 1-motives $[\mathbb{Z}^n \to \mathbb{G}_m^n]$, this amounts to Schanuel's conjecture: if $x_1, \ldots, x_n \in \mathbb{C}$ are $\mathbb{Q}$-linearly independent,

$$\mathrm{TrDeg}_{\mathbb{Q}}\, \mathbb{Q}[x_1, \ldots, x_n, e^{x_1}, \ldots, e^{x_n}] \geqslant n.$$

Interlude: motivic Galois groups and specializations problems.

$X \to S$ family of projective smooth varieties $s \in S \rightsquigarrow X_s \rightsquigarrow G_{X_s}$

Variation of G_{X_s} with s?

e.g. non-constant family of elliptic curves: $G_{X_s} = GL_2$ if X_s non CM.

General result: *Outside a countable union S_{ex} of algebraic subvarieties $S_n \subsetneq S$, G_{X_s} is (locally) constant. If the family is defined over $\bar{\mathbb{Q}}$, $S_{ex}(\bar{\mathbb{Q}}) \neq S(\bar{\mathbb{Q}})$.* (A. '96)

Application: $H^2(X_s)^{G_{X_s}} = NS_{X_s}$. Thus:

NS_{X_s} is constant outside S_{ex}; if the family is defined over $\bar{\mathbb{Q}}$, there exists $s \in S(\bar{\mathbb{Q}})$ such that NS specializes isomorphically at s.

(Similarly, if G is the motivic Galois of some complex abelian variety, there is an abelian variety A defined over $\bar{\mathbb{Q}}$ with $G_A = G$.)

Hint: X_s defined over $K \rightsquigarrow \rho_{X_s} : G_K \to G_{X_s}(\mathbb{Q}_\ell) \subset GL(H_{et}(X_{s,\bar{K}}, \mathbb{Q}_\ell))$.

Conjecturally, Im ρ_{X_s} Zariski dense; thus if $\eta \rightsquigarrow s$ is a specialization and G_{X_s} is smaller than G_{X_η}, then Im ρ_{X_s} is smaller than Im ρ_{X_η}. But this can be proved unconditionally. Conclude by Hilbert irreducibility argument (Serre's "infinite" variant).

Refinement (Cadoret - Tamagawa): when S is a curve defined over a number field, the set of points of S_{ex} of bounded degree is finite.

Similar situation with periods instead of Galois representations:

Conjecturally (**PC**), Im $(\operatorname{Spec}\mathbb{C} \overset{\varpi_{X_s}}{\to} \Pi_{X_s})$ Zariski-dense. Thus if $\eta \rightsquigarrow s$ is a specialization and G_{X_s} is smaller than G_{X_η}, then Im ϖ_{X_s} is smaller than Im ϖ_{X_η}. But this can be proved unconditionally (one of the threads which led me to the AO conjecture...)

Outline of the **AO** conjecture.

Geometry of $\mathcal{A}_g$, the algebraic variety which parametrizes principally polarized abelian varieties of dimension g (e.g. $\mathcal{A}_1 = j$-line).

Special subvarieties of $\mathcal{A}_g$: subvarieties which parametrize PPAV with "extra symmetries".

PPAV with maximal symmetry (complex multiplication) are parametrized by *special points*.

AO conjecture: *special subvarieties of* $\mathcal{A}_g$ *are characterized by the density of their special points*.

Remarks. - $\mathcal{A}_g$ and its special subvarieties share a common geometric nature: they are *Shimura varieties*

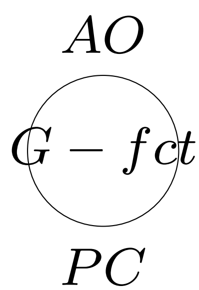
$$g = 1: \qquad \begin{array}{c} \mathbb{C} \\ \downarrow \wp \\ E \cong \mathbb{C}/(\mathbb{Z}\omega_1 + \mathbb{Z}\omega_2) \end{array} \qquad \begin{array}{c} \mathfrak{H} \\ \downarrow j \\ \mathbb{A}^1 \cong \mathfrak{H}/SL_2(\mathbb{Z}) \end{array}$$

- "extra symmetries" ? ... Prescribed endomorphisms on A, or more generally, prescribed Hodge cycles on powers of A; looks transcendental, but is an algebraic condition: amounts to prescribe algebraic cycles on product of powers of A and some compact abelian pencils (A. '96).

The AO conjecture (for $\mathcal{A}_g$) is now a *theorem* (2015), after two decades of collaborative efforts putting together many different areas. Some key contributors: A. Yafaev, E. Ullmo, B. Klingler, J. Pila, J. Tsimerman...

Connections between **AO** and **PC**.

1. Early circle of ideas which gave rise to the AO conjecture.



G-functions $\leftrightarrow$ periods

Variational approach to **PC** (for abelian periods) through G-functions?

Example. $E_\lambda : \ y^2 = x(x-1)(x-\lambda)$,

$$\omega_1(\lambda) \sim \pi F(\lambda), \eta_1 \sim \pi F'(\lambda), \quad F = F(\tfrac{1}{2}, \tfrac{1}{2}, 1; \lambda).$$

Diophantine theory of special values of G-functions $F, F' \rightsquigarrow$ *new proof of* **PC** *for CM elliptic curves* (A. ('96)).

For λ singular modulus (ie special point), $F(\lambda)(F'(\lambda)+\alpha F(\lambda)) = \beta/\pi$, $\alpha, \beta \in \bar{\mathbb{Q}}$. One cannot eliminate π... other solutions of the HGE are useless (log singularity at 0). But for AV of dim. $g > 1$ instead parametrized by a curve in $\mathcal{A}_g$ (instead of λ-line), one may get enough G-functions and relations between their special values.

Existence of lots of special points on the curve would allow to apply G-function theory efficiently. But analogy with Manin-Mumford renders the existence of ∞ly many special points unlikely in the non-modular case!

This was one source of my formulation of **AO** ('89) (Oort's later but independent formulation came from another source: CM liftings, Coleman conjecture...).

AO bounds the hope for an application of G-fct. theory to **PC**; nevertheless, more intricate alternative connections between **AO** and **PC** exist.

2. Curves in products of modular curves

Case of $C \subset \mathbb{A}^1 \times \mathbb{A}^1 \subset \mathcal{A}_2$ (A. ('93) - first, and only, unconditional case of **AO**, until Pila (2011)):

AO$_{\mathbb{A}^1 \times \mathbb{A}^1}$: *if* C *contains* ∞*ly many pairs of singular moduli* (j, j')*,* C *is either a vertical or horizontal line or some* $X_0(N)$*.*

i) (j_n, j'_n) singular moduli on C, (D_n, D'_n) (discriminants of quadratic orders). Class field theory $\rightsquigarrow$ for $n >> 0$, $\mathbb{Q}(\sqrt{D_n}) = \mathbb{Q}(\sqrt{D'_n})$ and D'_n/D_n takes finitely many values.

ii) Linear forms in elliptic periods $\rightsquigarrow$ if ∞ly many special points on C, a branch of C goes to (∞, ∞): if $(j_n = j(\tau_n), j'_n) \rightarrow (\infty, j' = j(\tau'))$, then $\tau' = \omega'_1/\omega'_2$ is well-approximated by quadratic numbers τ_n; contradicts Masser's lower bound for $|\omega'_1 - \tau_n \omega'_2|$.]

iii) analysis of Puiseux expansions.

3. Hypergeometric values

$a, b, c \in \mathbb{Q}$, $\Re(c) > \Re(b) > 0$, $n = \mathrm{den}(a, b, c)$,

$$F(a, b, c; \lambda) = \sum \frac{(a)_m (b)_m}{(c)_m\, m!} \lambda^m$$

$$= \frac{\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-\lambda x)^{-a} dx}{B(b, c-b)}$$

satisfies HG diff. equation, monodromy = Schwarz triangle group Δ.

numerator = period of $J^{new}_{n,a,b,c,\lambda}$

$$y^n = x^{n(b-1)}(1-x)^{n(c-b-1)}(1-\lambda x)^{-na}$$

denominator $B(b, c-b)$ = period of simple CM quotient $F_{b,c}$ of Fermat jacobian.

Question (J. Wolfart): *for which* (a,b,c) *are there* ∞*ly many* $\lambda \in \bar{\mathbb{Q}}$ *with* $F(a,b,c;\lambda) \in \bar{\mathbb{Q}}$*?*

Answer (Wüstholz-Wolfart-Cohen-Edixhoven - Yafaev): *iff* Δ *finite or arithmetic.*

[“if” due to Wolfart. “Only if”: 3 steps:

i) Wüstholz (special case of **PC**): $\bar{\mathbb{Q}}$-*linear* relations between periods of abelian periods come from endomorphisms

$\rightsquigarrow (\lambda, F(a,b,c;\lambda) \in \bar{\mathbb{Q}}) \Rightarrow J^{new}_{n,a,b,c,\lambda} \sim F_{b,c}$,

ii) for $\mathbb{P}^1 \backslash \{0,1,\infty\} \xrightarrow{\phi} \mathcal{A}_g : \lambda \mapsto J^{new}_{n,a,b,c,\lambda}$,

$Im(\phi)$ special iff Δ finite or arithmetic.

iii) **AO** $\rightsquigarrow J^{new}_{n,a,b,c,\lambda}$ has CM for ∞ly many λ's iff $Im(\phi)$ special.]

4. Bialgebraicity

$\mathfrak{H}_g \quad \subset \mathfrak{H}_g^\vee$ (lagrangian grassmannian)

$\mathrm{j} \downarrow \qquad\qquad\qquad \tau = \Omega_1 \Omega_2^{-1} \mapsto \mathrm{j}(\tau)$

$\mathcal{A}_g$

Both $\mathfrak{H}_g^\vee$ and $\mathcal{A}_g$ are algebraic varieties/$\mathbb{Q}$, but j is transcendental.

Bialgebraic characterization of special subvarieties (Wüstholz-Cohen-Shiga-Wolfart-Ullmo-Yafaev): *$S \subset \mathcal{A}_g$ is special iff both S and a branch of $\mathrm{j}^{-1}(S) \subset \mathfrak{H}_g^\vee$ are algebraic and defined over $\bar{\mathbb{Q}}$.*

CSW: case of dim 0: $\tau, \mathrm{j}(\tau) \in \bar{\mathbb{Q}} \Leftrightarrow \mathrm{j}(\tau)$ is a special point (Schneider).

Remark. $\mathfrak{H}_g^\vee$ and $\mathcal{A}_g$ are *transcendentally* related by $\mathfrak{j}$, but are also *algebraically* related by the relative period torsor:

$$\begin{array}{ccc} \Pi_g & \xrightarrow{\rho} & \mathfrak{H}_g^\vee \\ \downarrow & & \\ \mathcal{A}_g & & \end{array}$$

Π_g is a Sp_{2g}-torsor on $\mathcal{A}_g$, with coordinates corresponding to $\begin{pmatrix} \Omega_1 & N_1 \\ \Omega_2 & N_2 \end{pmatrix}$, and ρ is the Sp_{2g}-equivariant surjective map $\begin{pmatrix} \Omega_1 & N_1 \\ \Omega_2 & N_2 \end{pmatrix} \mapsto \Omega_1 \Omega_2^{-1}$.

5. Minimal special subvarieties

Given a PPAV A of dim. g, ie a point $\mathrm{j}_A \in \mathcal{A}_g$, there is a (unique) minimal special subvariety S_A containing j_A.

Question (Wolfart): *if A is defined over $\bar{\mathbb{Q}}$, what is the dimension of S_A?*

Answer: $\mathbf{PC}_A \Rightarrow \dim S_A = \mathrm{TrDeg}._{\mathbb{Q}}\,\mathbb{Q}(\tau)$, for any $\tau \in \mathfrak{H}$ such that $\mathrm{j}(\tau) = \mathrm{j}_A$).

via an analysis of (a reduction of) the relative period torsor Π_g. CSW is the case '0= 0'' of this equality.

An afterthought.

One breakthrough in the proof of **AO** was the introduction of o-minimal methods (Pila-Zannier). In such "counting arguments", an exceptional (semi-)algebraic set is left out. To handle it, one needs some functional transcendence results, which are functional analogs of the generalized **PC**.

Very recently, the Pila-Zannier method led to very powerful functional transcendence results ("Ax-Schanuel").

Question: *is there any way to* use *the Pila-Zannier method in transcendental* **Number** *Theory?*

For instance, Schneider's problem:

prove that

"$\tau, \mathrm{j}(\tau) \in \bar{\mathbb{Q}} \Leftrightarrow \mathrm{j}(\tau)$ *is a special point*"

using only the j-function (no $\wp$!)

is still open... can "counting arguments" à la Pila-Wilkie help in this (and other similar) context(s)?