Answer to some question by Fujita on Variation of Hodge Structures (the direct image of $\omega$ needs not be semiample).

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Outline

1. Fujita’s theorems
2. Answer to Fujita’s question
3. Hermitian curvature
4. Sketch of proof of Fujita’s theorem
5. Hypergeometric integrals leading to a unitary flat bundle $Q$ of infinite order
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Fujita’s first theorem

An important progress in classification theory was stimulated by a theorem of Fujita, who showed (On Kähler fiber spaces over curves, J. Math. Soc. Japan 30 (1978), no. 4, 779–794):

**Theorem**

If $X$ is a compact Kähler manifold and $f: X \to B$ is a fibration onto a projective curve $B$ (i.e., $f$ has connected fibres), then the direct image sheaf

$$V := f_* \omega_X|_B = f_* (\mathcal{O}_X(K_X - f^* K_B))$$

is a nef vector bundle on $B$. This means that each quotient bundle $Q$ of $V$ has degree $\deg(Q) \geq 0$; sometimes, instead of the word nef, one uses the terminology ‘$V$ is numerically semipositive’.
Here, $K_X$ is the Cartier divisor of the invertible sheaf $\Omega^n_X$ of holomorphic $n$-forms, where $n = \text{dim}_\mathbb{C}(X)$, and we have a vector bundle

$$V := f_*\omega_X|_B = f_* (\mathcal{O}_X(K_X - f^*K_B)).$$

The fibre of $V$ over a point $b \in B$ such that $X_b := f^{-1}(b)$ is smooth is then the vector space of holomorphic $(n - 1)$-forms on $X_b$,

$$V_b = H^0(X_b, \Omega^{n-1}_X).$$
Kawamata’s theorem

Soon afterwards, using Griffiths’ results on Variation of Hodge Structures, Kawamata improved on Fujita’s result, solving a long standing problem and proving the subadditivity of Kodaira dimension for such fibrations,

\[ \text{Kod}(X) \geq \text{Kod}(B) + \text{Kod}(F), \]

(here \(F\) is a general fibre) showing the semipositivity also for the direct image of higher powers of the relative dualizing sheaf

\[ W_m := f_*(\omega_X^m|_B) = f_*(\mathcal{O}_X(m(K_X - f^*K_B))). \]

Much later, Kawamata extended his result to the case where the dimension of the base variety \(B\) is \(> 1\).
Fujita’s second theorem

In the note *The sheaf of relative canonical forms of a Kähler fiber space over a curve* Proc. Japan Acad. Ser. A Math. Sci. 54 (1978), no. 7, 183–184, Fujita announced the following stronger result, sketching the idea of proof, but referring to a forthcoming article concerning the positivity of the so-called local exponents (this article was never written).

**Theorem**

*(Fujita’s second theorem)*

*Let* $f : X \to B$ *be a fibration of a compact Kähler manifold* $X$ *over a projective curve* $B$, *and consider the direct image sheaf*

$$V := f_*\omega_{X|B} = f_*\left(\mathcal{O}_X(K_X - f^*K_B)\right).$$

*Then* $V$ *splits as a direct sum* $V = A \oplus Q$, *where* $A$ *is an ample vector bundle and* $Q$ *is a unitary flat bundle.*
Ample, semiample, nef

Let $V$ be a holomorphic vector bundle over a projective curve $B$.

**Definition**

Let $p : \mathbb{P} := \text{Proj}(V) = \mathbb{P}(V^\vee) \to B$ be the associated projective bundle, and let $H$ be a hyperplane divisor (s.t. $p_*(\mathcal{O}_\mathbb{P}(H)) = V$). Then $V$ is said to be:

- (NP) numerically semi-positive if and only if every quotient bundle $Q$ of $V$ has degree $\deg(Q) \geq 0$,
- (NEF) nef if and only if $H$ is nef on $\mathbb{P}$ ($H \cdot C \geq 0$ for each curve $C \subset \mathbb{P}$),
- (A) ample if and only if $H$ is ample on $\mathbb{P}$ ($|mH|$ yields an embedding of $\mathbb{P}$ for $m \gg 0$),
- (SA) semi-ample if and only if $H$ is semi-ample on $\mathbb{P}$ (there is a positive multiple $mH$ yielding a morphism).

Recall that $(A) \Rightarrow (SA) \Rightarrow (NEF) \iff (NP)$. 
Flat and unitary flat bundles

Definition

A flat holomorphic vector bundle on a complex manifold $M$ is a holomorphic vector bundle $\mathcal{H} := \mathcal{O}_M \otimes_{\mathbb{C}} \mathbb{H}$, where $\mathbb{H}$ is a local system of complex vector spaces associated to a representation $\rho : \pi_1(M) \to GL(r, \mathbb{C})$,

$$\mathbb{H} := (\tilde{M} \times \mathbb{C}^r)/\pi_1(M),$$

$\tilde{M}$ being the universal cover of $M$ (so that $M = \tilde{M}/\pi_1(M)$).

We say that $\mathcal{H}$ is unitary flat if it is associated to a representation $\rho : \pi_1(M) \to U(r, \mathbb{C})$. 

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Fujita’s question

Recall Fujita’s second theorem, for which a complete proof was given in our joint work with Michael Dettweiler (arXiv 1311.3232 and CRAS Ser. I, 352 (2014), 241-244)

**Theorem (Fujita’s second theorem)**

Let $f : X \rightarrow B$ be a fibration of a compact Kähler manifold $X$ over a projective curve $B$. Then

$$V := f_\ast \omega_X|_B = f_\ast (\mathcal{O}_X(K_X - f_\ast K_B))$$

splits as $V = A \oplus Q$, with $A$ an ample vector bundle and $Q$ a unitary flat bundle.

Fujita posed in 1982 (Proceedings of the 1982 Taniguchi Conference) the following

**Question (Fujita)** Is the direct image $V := f_\ast \omega_X|_B$ semi-ample?
Fujita’s theorem and Fujita’s question

The following result is due to Hartshorne:

**Proposition**

A vector bundle $V$ on a curve is nef if and only if it is numerically semi-positive, i.e., if and only if every quotient bundle $Q$ of $V$ has degree $\deg(Q) \geq 0$, and $V$ is ample if and only if every quotient bundle $Q$ of $V$ has degree $\deg(Q) > 0$.

Then there is a technical result we established, which clarifies how Fujita’s question is related to Fujita’s II theorem

**Theorem**

Let $\mathcal{H}$ be a unitary flat vector bundle on a projective manifold $M$, associated to a representation $\rho : \pi_1(M) \to U(r, \mathbb{C})$. Then $\mathcal{H}$ is nef and moreover $\mathcal{H}$ is semi-ample if and only if $\text{Im}(\rho)$ is finite.
Answer to Fujita’s question

This is the main new result in our joint work with Dettweiler:

**Theorem**

There exist surfaces $X$ of general type endowed with a fibration $f : X \to B$ onto a curve $B$ of genus $\geq 3$, and with fibres of genus 6, such that $V := f_*\omega_X|_B$ splits as a direct sum $V = A \oplus Q_1 \oplus Q_2$, where $A$ is an ample rank-2 vector bundle, and the flat unitary rank-2 summands $Q_1, Q_2$ have infinite monodromy group (i.e., the image of $\rho_j$ is infinite). In particular, $V$ is not semi-ample.

Thus Fujita’s answer has a negative answer in general.
Cases where $V$ is semiample.

Corollary

Let $f : X \rightarrow B$ be a fibration of a compact Kähler manifold $X$ over a projective curve $B$. Then $V := f_*\omega_X|_B$ is a direct sum $V = A \bigoplus (\bigoplus_{i=1}^h Q_i)$, with $A$ ample and each $Q_i$ unitary flat without any nontrivial degree zero quotient. Moreover,

(I) if $Q_i$ has rank equal to 1, then it is a torsion bundle ($\exists$ $m$ such that $Q_i^\otimes m$ is trivial) (Deligne)

(II) if the curve $B$ has genus 1, then $\text{rank } (Q_i) = 1$, $\forall i$.

(III) In particular, if $B$ has genus at most 1, then $V$ is semi-ample.

(I) This was proven by Deligne (and by Simpson using the theorem of Gelfond-Schneider)

(II) Follows since $\pi_1(B)$ is abelian, if $B$ has genus 1: hence every representation splits as a direct sum of 1-dimensional ones.
Flat versus unitary flat

While a unitary flat bundle is nef, the same does not hold for a flat bundle.

**Theorem (C-Dettweiler)** Let $f : X \to B$ be a Kodaira fibration, i.e., $X$ is a surface and all the fibres of $f$ are smooth curves of genus $g \geq 2$ not all isomorphic to each other. Then $V := f_*\omega_{X|B}$ has strictly positive degree, hence $\mathcal{H} := R^1f_*(\mathcal{C}) \otimes \mathcal{O}_B$ is a flat bundle which is not nef.

**Proof** 1) Since all the fibres of $f$ are smooth, $V = f_*(\Omega^1_X|_B)$ and we have an exact sequence

$$0 \to V \to \mathcal{H} \to V^\vee \to 0,$$

and it suffices to show that the degree of the quotient bundle $V^\vee$ is strictly negative, or, equivalently, $\deg(V) > 0$. 
Flat versus unitary flat, cont.

We want to show that \( \deg(V) > 0 \).
We have that (if \( X \) is minimal)

\[
\deg(V) = K_X^2 - 8(b - 1)(g - 1),
\]

where \( g \) is the genus of the fibres of \( f \), and \( b \) is the genus of \( B \). As well known also the genus \( b \geq 2 \), and therefore \( X \) contains no rational curve and is therefore a minimal surface.
Since \( f \) is a differentiable fibre bundle, we have for the Euler-Poincaré characteristic of \( X \)

\[
e(X) = 4(b - 1)(g - 1).
\]

Kodaira proved that for such fibrations the topological index \( \sigma(X) \), the signature of the intersection form on \( H^2(X, \mathbb{R}) \) is positive. By the index theorem we have

\[
0 < 3\sigma(X) = c_1^2(X) - 2c_2(X) = K_X^2 - 2e(X) = \deg(V).
\]
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Curvature decreases in subbundles?

The example of Kodaira fibrations produces subbundles of a flat bundle (they have zero curvature) which are positively curved. This contradicts the slogan above? Not really, the correct one is (see the book by Griffiths and Harris):

**curvature decreases in Hermitian subbundles.** The above is the first ingredient in the proof of the theorem mentioned above.

**Theorem**

Let $\mathcal{H}$ be a unitary flat vector bundle on a projective manifold $M$, associated to a representation $\rho : \pi_1(M) \to U(r, \mathbb{C})$. Then $\mathcal{H}$ is nef and moreover $\mathcal{H}$ is semi-ample if and only if $\text{Im}(\rho)$ is finite.

Since $\mathcal{H}$ is unitary flat, $\mathcal{H}$ is a Hermitian holomorphic bundle, and by the principle ‘curvature decreases in Hermitian subbundles’ each subbundle has degree $\leq 0$ and each quotient bundle $W$ of $\mathcal{H}$ has degree $\geq 0$, hence $\mathcal{H}$ is nef.
Unitary flat bundles

If $\mathcal{H}$ is unitary flat, we saw that $\mathcal{H}$ is a Hermitian holomorphic bundle, and by the principle ‘curvature decreases in Hermitian subbundles’ each subbundle has degree $\leq 0$ and each quotient bundle $\mathcal{W}$ of $\mathcal{H}$ has degree $\geq 0$, hence $\mathcal{H}$ is nef. Moreover, by Lefschetz’ theorem, we can reduce to the case where $M$ is a curve. Let $B$ be a projective curve and $\rho : \pi_1(B) \to U(r, \mathbb{C})$ a unitary representation, and $\mathcal{H}_\rho$ the associated flat holomorphic bundle. Since $\rho$ is unitary, it is a direct sum of irreducible unitary representations $\rho_j, j = 1, \ldots k$. Accordingly, we have a splitting

$$\mathcal{H}_\rho = \bigoplus_{j=1}^{k} \mathcal{H}_{\rho_j}.$$ 

Narasimhan and Seshadri have proven that each $\mathcal{H}_{\rho_j}$ is a stable degree zero holomorphic bundle on $B$. This result plays another crucial role for the proof of the above theorem.
Curvature and numerical positivity

Definition

Let \((E, h)\) be an Hermitian vector bundle on a complex manifold \(M\). Take the canonical Chern connection associated to the Hermitian metric \(h\), and denote by \(\Theta(E, h)\) the associated Hermitian curvature, which gives a Hermitian form on the complex vector bundle bundle \(T_M \otimes E\).

Then one says that \(E\) is Nakano positive (resp.: semi-positive) if there exists a Hermitian metric \(h\) such that the Hermitian form associated to \(\Theta(E, h)\) is strictly positive definite (resp.: semi-positive definite).

Remark

Umemura proved that a vector bundle \(V\) over a curve \(B\) is positive (i.e., Griffiths positive, or equivalently Nakano positive) if and only if \(V\) is ample.
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Idea of proof in the case of no singular fibres

$V$ is a holomorphic subbundle of the holomorphic vector bundle $\mathcal{H}$ associated to the local system $\mathbb{H} := \mathcal{R}^m f_*(\mathbb{Z}_X)$, i.e.,

$\mathcal{H} = \mathbb{H} \otimes_{\mathbb{Z}} \mathcal{O}_B$.

The bundle $\mathcal{H}$ is flat, hence the curvature $\Theta_{\mathcal{H}}$ associated to the flat connection satisfies $\Theta_{\mathcal{H}} \equiv 0$.

We view $V$ as a holomorphic subbundle of $\mathcal{H}$, while

$$V^\vee \cong R^m f_* \mathcal{O}_X, \quad m = \dim(X) - 1$$

is a holomorphic quotient bundle of $\mathcal{H}$.

By the curvature formula for subbundles

$$\Theta_V = \Theta_{\mathcal{H}}|_V + \bar{\sigma}^t \sigma = \bar{\sigma}^t \sigma,$$

Griffihts proves that the curvature of $V^\vee$ is semi-negative, since its local expression is of the form $i h'(z) d\bar{z} \wedge dz$, where $h'(z)$ is a semi-positive definite Hermitian matrix.
The case of no singular fibres

In particular we have that the curvature $\Theta_V$ of $V$ is semipositive and, moreover, that the curvature vanishes identically if and only if the second fundamental form $\sigma$ vanishes identically, i.e., if and only if $V$ is a flat subbundle. However, by semi-positivity, we get that the curvature vanishes identically if and only its integral, the degree of $V$, equals zero. Hence $V$ is a flat bundle if and only if it has degree 0. The same result then holds true, by a similar reasoning, for each holomorphic quotient bundle $Q$. 
Answer to some question by Fujita on VHS
Sketch of proof of Fujita’s theorem
The general case

In the general case we use:
1) The semistable reduction theorem (a base change $B' \to B$ such that all fibres of the pull-back $X' \to B'$ are reduced with normal crossings)
2) Comparing the pull-back of $V$ with the analogously defined $V'$
3) Some crucial estimates given by Zucker (using Schmid’s asymptotics for Hodge structures) for the growth of the norm of sections of the $L^2$-extension of Hodge bundles.
The general case, cont.

4) A lemma by Kawamata

**Lemma**

*Let $L$ be a holomorphic line bundle over a projective curve $B$, and assume that $L$ admits a singular metric $h$ which is regular outside of a finite set $S$ and has at most logarithmic growth at the points $p \in S$. Then the first Chern form $c_1(L, h) := \Theta_h$ is integrable on $B$, and its integral equals $\deg(L)$.*

This shows that in the semistable case singularities are ininfluent, and the argument runs as in the case of no singular fibres.
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Symmetry by a cyclic group of order 7

**Proposition**

Let \( f : X \to B \) be a semistable fibration of a surface \( X \) onto a projective curve, such that the group \( G = \mu_7 \cong \mathbb{Z}/7 \) acts on this fibration inducing the identity on \( B \). Assume that the general fibre \( F \) has genus 6 and that \( G \) has exactly 4 fixed points on \( F \), with tangential characters \( (1, 1, 1, 4) \).

Then if we split \( V = f^*(\omega_X|_B) \) into eigensheaves, then the eigensheaves \( V_1, V_2 \) are unitary flat rank 2 bundles.
Symmetry by a cyclic group of order 7

Proposition

Let $f : X \to B$ be a semistable fibration of a surface $X$ onto a projective curve, such that the group $G = \mu_7 \cong \mathbb{Z}/7$ acts on this fibration inducing the identity on $B$. Assume that the general fibre $F$ has genus 6 and that $G$ has exactly 4 fixed points on $F$, with tangential characters $(1, 1, 1, 4)$. Then if we split $V = f_*(\omega_X|_B)$ into eigensheaves, then the eigensheaves $V_1, V_2$ are unitary flat rank 2 bundles.

Since the fibration is semistable, the local monodromies are unipotent: on the other hand, they are unitary, hence they must be trivial. This implies that the local systems $\mathbb{H}_1^*$ and $\mathbb{H}_2^*$ have respective flat extensions to local systems $\mathbb{H}_1$ and $\mathbb{H}_2$ on the whole curve $B$. 
Symmetry by a cyclic group of order 7, cont.

Denote by $\mathcal{H}_j := \mathbb{H}_j \otimes \mathcal{O}_B$, $j = 1, 2$. Direct calculation shows that $V_j = \mathcal{H}_j$ over $B^* = B \setminus S$, $S$ being the set of critical values of $f$. We saw that the norm of a local frame of $V_j$ has at most logarithmic grow at the points $p \in S$. This shows that $V_j$ is a subsheaf of $\mathcal{H}_j$: by semipositivity we conclude that we have equality $V_j = \mathcal{H}_j$. 
The examples

The equation

$$z_1^7 = y_1 y_0 (y_1 - y_0)(x_0 y_1 - x_1 y_0)^4 x_0^3.$$ 

describes a singular surface $\Sigma'$ which is a cyclic covering of $\mathbb{P}^1 \times \mathbb{P}^1$ with group $G := \mathbb{Z}/7$.

Let $Y$ be a minimal resolution of singularities of $\Sigma$: $Y$ admits a fibration $\varphi: Y \to \mathbb{P}^1$ with fibres curves of genus 6.

We let $X$ be the minimal resolution of the fibre product of $\varphi: Y \to \mathbb{P}^1$ with $\psi: B \to \mathbb{P}^1$, where $\psi$ is the $G$-Galois cover branched on $\infty = \{x_0 = 0\}$, $0 = \{x_1 = 0\}$, $1 = \{x_1 = x_0\}$, and with local characters $(1, 1, -2)$. In particular $B$ has genus 3 by Hurwitz’ formula.
Properties of the example

**Theorem**

The above surface $X$ is a surface of general type endowed with a fibration $f : X \to B$ onto a curve $B$ of genus 3, and with fibres of genus 6, such that $V := f^* \omega_X |_B$ splits as a direct sum $V = A \oplus Q_1 \oplus Q_2$, where $A$ is an ample rank-2 vector bundle, and the unitary flat rank-2 summands $Q_1, Q_2$ have infinite monodromy.

The last assertion is a consequence of the classification by Schwarz of the cases where the monodromy of hypergeometric integrals is finite.
Hypergeometric integrals

Another example is given by the equation

\[ z_1^7 = y_1 y_0^4 (y_1 - y_0)(y_1 - xy_0), \quad x \in \mathbb{C} \setminus \{0, 1\} \]

which gives another family of curves. It is similar to the previous family, except that we get here \( V_1 \) generated by

\[ \eta := y^{-\frac{6}{7}} (y - 1)^{-\frac{6}{7}} (y - x)^{-\frac{6}{7}} dy, \text{ and by } y \cdot \eta. \]

Varying \( x \), we obtain a rank-2 local system over \( \mathbb{P}^1 \setminus \{0, 1, \infty\} \), which is equivalent, in view of the Riemann-Hilbert correspondence, to a second order differential equation with regular singular points. Indeed, using results of Deligne-Mostow and Kohno, we see that we have a Gauss hypergeometric equation, and we can see that the local monodromies have order 7, hence we are not in the Schwarz list and the monodromy is infinite (and irreducible).