Colmez Conjecture in Average

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Assume $\mathscr{A}$ is semiabelian, then height is invariant under base change.

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(2) $[E:\mathbb{Q}] = 4$ by Tonghai Yang.

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$\eta_{E/F}$: the corresponding quadratic character of $\mathbb{A}_F^\times$.
$L_f(s,\eta)$: the finite part of the completed L-function $L(s,\eta)$.

Theorem (Xinyi Yuan –)

$$\frac{1}{2^g}\sum_{\Phi} h(\Phi) = -\frac{1}{2}\frac{L_f'(\eta_{E/F},0)}{L_f(\eta_{E/F},0)} - \frac{1}{4}\log(d_{E/F}d_F).$$

where Φ runs through the set of CM types E.

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Our proof is different than theirs: we use neither high dimensional Shimura varieties nor Borcherds' liftings.

Ideal of proof: $g = 1$

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$$h(A) = \frac{1}{12[K:\mathbb{Q}]} \left(\log|\mathrm{disc}(A)| - \sum_{\sigma:K\to\mathbb{C}} \log\left|\eta(q_\sigma)^{24}(4\pi \mathrm{Im}\tau_\sigma)^6\right| \right).$$

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When A has CM, apply either Kronecker–Limit or Chowla–Selberg formula.

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$$\overline{T}(q) = \overline{T}_0 \left(1 - \frac{3}{\pi y}\right) + \sum \overline{T}_n q^n, \qquad \overline{T}_0 = -\pi_1^* \overline{\omega} - \pi_2^* \overline{\omega}.$$

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This series is proportional the Eisenstein series of weight 2:

$$E_2(\tau) = -\frac{1}{24}\left(1 - \frac{3}{\pi y}\right) + \sum_n \sigma_1(n) q^n.$$

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Theorem

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May replace LHS by $\frac{1}{2}h(A_0, \tau)$ for an abelian variety A_0 with action by O_E and isogenous to $A_{\Phi_1} + A_{\Phi_2}$. Such an A_0 corresponds to a CM-point Q in a unitary Shimura curve Y.

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- $\kappa : O_{\mathbb{B}} \longrightarrow \widehat{T}(A)$, a class of homomorphism of $\widehat{O}_E$-modules such that the symplectic form ψ_θ on $\widehat{T}(A)$ is given by

$$\psi_\theta(\kappa x, \kappa y) = \mathrm{tr}_{\mathbb{B}_f/\widehat{\mathbb{Z}}}(\sqrt{\lambda} x \bar{y}).$$

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The direct methods of extending the moduli problem to $O_{E^\natural}$ usually do not yield a regular integral scheme.
We will construct integral models using quaternionic Shimura curve X over F, where the regular integral models have been constructed by Carayol, and Čerednik–Drinfeld.

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$$M(v) \subset H_1^{dR}(v) \xrightarrow{\nabla} H_1^{dR}(v) \otimes \Omega_X^1.$$

2. a local system T of free $O_{\mathbb{B}}$-module of rank 1 in étale topology.

In this way we have étale sheaf of torsion $O_{\mathbb{B}}$-modules:

$$G = (T \otimes_{\widehat{\mathbb{Z}}} \widehat{\mathbb{Q}})/T = \bigoplus_{\wp} G_{\wp}$$

where the sum runs over the set of finite places $\wp$ of F, and $G_{\wp}$ is an étale sheaf of torsion $O_{\mathbb{B},\wp}$-modules .

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$$G = f^* A_{I,\mathrm{tor}}, \qquad G_\wp = f^* A_I[\wp^\infty].$$

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In this way we obtain an extension of the bundle N on $\mathscr{X}_U$:

$$\mathscr{N}(\wp) = \det \mathscr{M}(\wp) \otimes \det \mathscr{M}^\vee(\wp).$$

This bundles also has metrics at archimedean places by Hodge structures.

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where D_B is the ramification divisor on $\mathrm{Spec}\, O_F$ of $\mathbb{B}$.
In summary, the calculation of $h(\Phi_1, \Phi_2) = \frac{1}{2}h(A_0, \tau)$ is reduced to the calculation of $h_{\overline{\omega}_{\mathscr{X}}}(P)$ at a special point P on $\mathscr{X}$.