Algebraic flows on Abelian Varieties and Shimura Varieties.

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Abelian Ax-Lindemann

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Abelian Ax-Lindemann theorem, (Pila-Zannier)

Theorem

Let Θ be an irreducible algebraic subvariety of $\mathbb{C}^g$. The Zariski closure of $\pi(\Theta)$ is weakly special.

Question

What can be said about the topological closure $\overline{\pi(\Theta)}$ of $\pi(\Theta)$?

Real weakly special subvarieties

Definition

Let $W \subset \mathbb{C}^g$ be a $\mathbb{R}$-vector space such that $\Gamma_W := \Gamma \cap W$ is a lattice in W. Then W/Γ_W is a real torus and is a closed real analytic subset of A.

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Conjecture

Let A be a abelian variety of dimension g. Let Θ be a complex irreducible algebraic subvariety of $\mathbb{C}^g$. Then there exists a finite number $Z_1, \ldots, Z_r$ of real weakly special subvarieties of A such that

$$\overline{\pi(\Theta)} = \pi(\Theta) \cup \bigcup_{k=1}^{r} Z_i.$$

Algebraic flows : Measure theoretic version

Let $\omega = \frac{i}{2}\sum_{k=1}^{g} dz_k \wedge d\overline{z}_k = \sum_{k=1}^{g} dx_k \wedge dy_k$, and for $R > 0$ let $B(0,R)$ be the complex ball

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Let Θ be an algebraic subvariety of $\mathbb{C}^g$ of dimension d. For all R big enough we define the probability measure $\mu_{\Theta,R}$ on $\mathbb{C}^g$ such that for any continuous function f on $\mathbb{C}^g$,

$$\mu_{\Theta,R}(f) = \frac{1}{V_R}\int_{\Theta\cap B(0,R)} f\omega^d$$

where $V_R = \int_{\Theta\cap B(0,R)} \omega^d$.

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$$\mu_{\Theta,R}(f) \to \sum_{k=1}^{r} c_k \mu_{Z_k}(f)$$

as $R \to \infty$.

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(i) We have

$$\overline{\pi(C)} = \pi(C) \cup \bigcup_{\alpha=1}^{r} \mathbb{T}_\alpha.$$

(ii) Let μ_α be the canonical probability measure on $\mathbb{T}_\alpha$. There exists positive real numbers $c_1, \ldots, c_r$ such that $\mu_{C,R}$ converges weakly to $\sum_{\alpha=1}^{r} c_\alpha \mu_\alpha$.

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$$\mathbb{T}_\Theta = \pi(P) + W_\Theta / \Gamma \cap W_\Theta.$$

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Then $\mathbb{T}_\Theta$ is independent of P and $\mathbb{T}_\Theta$ is the smallest real weakly special subvariety of A containing $\pi(\Theta)$. We say that $\mathbb{T}_\Theta$ is the **Mumford-Tate torus** *associated to Θ.*

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Remark

Let Θ be an irreducible complex algebraic subvariety of $\mathbb{C}^g$. Then $\overline{\pi(\Theta)} \subset \mathbb{T}_\Theta$.

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Remark

Let Θ be an irreducible complex algebraic subvariety of $\mathbb{C}^g$. Then $\overline{\pi(\Theta)} \subset \mathbb{T}_\Theta$. When do we have $\overline{\pi(\Theta)} = \mathbb{T}_\Theta$?

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Lemma

(a) $\overline{\pi(C_\alpha)} - C_\alpha \subset \mathbb{T}_\alpha$.

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Lemma

(a) $\overline{\pi(C_\alpha)} - C_\alpha \subset \mathbb{T}_\alpha$.
(b) $\mathbb{T}_\alpha \subset \mathbb{T}_C$.
(c) $\overline{\pi(C)} \subset \pi(C) \cup \bigcup_{\alpha=1}^r \mathbb{T}_\alpha$.

Let C_α be an infinite branch of C, $\mathbb{T}_\alpha$ be the associated asymptotic Mumford-Tate torus and μ_α be the canonical measure on $\mathbb{T}_\alpha$.

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The main result is a consequence of the previous lemma and

Theorem

$\mu_{\alpha,R}$ weakly converges to μ_α as R tends to ∞.

Example : The linear case.

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As a consequence $MT(W) \otimes \mathbb{R}/\Gamma \cap MT(W) \otimes \mathbb{R}$ is a real torus of real dimension 3.

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As a consequence $MT(W) \otimes \mathbb{R}/\Gamma \cap MT(W) \otimes \mathbb{R}$ is a real torus of real dimension 3.

This shows that we can't expect that in the conjecture 2 that the analytic closure of $\pi(W)$ has a complex structure.

Instructing example 1

Proposition

Let $n \geq 3$ be an integer. Let $C \in \mathbb{C}^2$ be the hyperelliptic curve with equation

$$Z_2^2 = Z_1^n + a_{n-1} Z_1^{n-1} + \cdots + a_0.$$

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In this case $\mathbb{T}_C = A$, with two branches C_1 near $(0, \infty)$ and C_2 near $(\infty, 0)$. Then $\mathbb{T}_1 = \{0\} \times E$ and $\mathbb{T}_2 = E \times \{0\}$.

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If $\Gamma \subset \mathbb{C}^2$ is such that the dual lattice $\widehat{\Gamma}$ of Γ contains no element of the form $(0, b)$ or of the form $(a, 0)$, then $\overline{\pi(C)} = \mathbb{C}^2/\Gamma = A$ and $\mu_{C,R} \to \mu_A$. In this case $\mathbb{T}_C = \mathbb{T}_1 = \mathbb{T}_2 = A$.

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Let C_α be an infinite branch of C. By Weyl's lemma the theorem is equivalent to showing that for all θ in $\widehat{\Gamma}$,

$$\mu_{\alpha,R}(\chi_\theta) \to \mu_\alpha(\chi_\theta).$$

Main results in terms of harmonic analysis.

Let ϕ be a function of a complex variable z with a Puiseux expansion given for z big enough by

$$\phi(z) = \sum_{n \geq 0} a_n z^{\alpha - \frac{n}{e}}$$

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Hyperbolic Ax-Lindemann

In this case $G = G^{ad}$, $\mathcal{D} = G(\mathbb{R})/K_\infty \subset \mathbb{C}^n$ is a bounded symmetric domain. Γ is an arithmetic lattices and $\pi : \mathcal{D} \to S = \Gamma \backslash \mathcal{D}$.

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Theorem

Hypebolic Ax-Lindemann, K-U-Y and P-T Let Z be an irreducible algebraic subvariety of $\mathcal{D}$. Then the Zariski closure of $\pi(Z)$ is weakly special.

Algebraic flows on hermitian locally symmetric spaces

Definition

A **real weakly special subvariety** *of S is a real analytic subset of S of the form*

$$Z = \Gamma \cap H(\mathbb{R})^+ \backslash H(\mathbb{R})^+ . x$$

where H is an algebraic subgroup of G such that the radical of H is unipotent and the real points of a $\mathbb{Q}$-simple factors of a Levi of H are not compact.

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Let Θ be an algebraic subvariety of $\mathcal{D}$.

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Product of two modular curves

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Let $G = \mathrm{SL}_2 \times \mathrm{SL}_2$, $X^+ = \mathbb{H} \times \mathbb{H}$ *and for some* $g \in \mathrm{SL}_2(\mathbb{R})$

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If $g \in G(\mathbb{Q})$, *the closure of* $\pi(Z)$ *is a special subvariety* $Y_0(n)$ *for some* $n \in \mathbb{N}$. *If* $g \notin G(\mathbb{Q})$, *then* $\pi(Z)$ *is dense in* $\Gamma \backslash X^+$.