Multiplicative relations among singular moduli

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Singular moduli

Mainly joint work with Jacob Tsimerman.

Singular moduli are the "special values" of the j-function.

Definition

A **singular modulus** is a complex number $j(\tau)$ where $j : \mathbb{H} \to \mathbb{C}$ is the modular function, and $\tau \in \mathbb{H}$ is quadratic ($[\mathbb{Q}(\tau) : \mathbb{Q}] = 2$).

$$\Sigma = \{\sigma = j(\tau) : \tau \in \mathbb{H}, [\mathbb{Q}(\tau) : \mathbb{Q}] = 2\}$$

(Schneider: These are the only points with $\tau, j(\tau) \in \overline{\mathbb{Q}}$.)

$j(\tau)$ is the j-invariant of $E_\tau = \Lambda_\tau$. These are the ell. cvs with CM.

Algebraic integers, and $[\mathbb{Q}(j(\tau)) : \mathbb{Q}] = \mathrm{Cl}(\mathcal{O}_{D(\tau)})$. Examples:

$j(\frac{1+\sqrt{-163}}{2}) = -2^{18}3^3 5^3 23^3 29^3$, $\quad j(\sqrt{-5}) = (50 + 26\sqrt{5})^3$.

André-Oort conjecture

For $\mathbb{C}^n = Y_1(\mathbb{C})^n$ as a Shimura variety (moduli of n-tuples of elliptic curves): Fix $V \subset \mathbb{C}^n$ and study points of $V \cap \Sigma^n$.

Special subvarieties in $\mathbb{C}^2$: points in Σ^2, vertical/ horizontal lines with fixed coord in Σ, modular curves $\Phi_N(x, y) = 0$, $\mathbb{C}^2$.

Theorem (André 1998; AO for $\mathbb{C}^2$)

A curve $V \subset \mathbb{C}^2$ containing infinitely many special points is special.

For $V \subset \mathbb{C}^n$, AO says that $V \cap \Sigma^n$ has a finite description in modular terms: "V contains only finitely many maximal special subvarieties"

Special subvarieties in $\mathbb{C}^n$: irreducible components of subvarieties defined by modular relations (any number) and setting coords to be fixed value in Σ (any number). Special points: Σ^n.

Multi-modular n-tuples

Definition

A **multi-modular** n**-tuple** is an n-tuple of distinct elements of Σ whose entries satisfy a non-trivial multiplicative relation, but such that no proper subset of them does.

Non-trivial mult relation: $\prod \sigma_i^{a_i} = 1, a_i \in \mathbb{Z}$ not all zero.

Example (A multi-modular 5-tuple)

$$-2^{15}3^35^311^3, \quad -2^{15}, \quad 2^33^311^3, \quad 2^63^3, \quad 2^{15}3^15^3$$

Theorem (+Jacob Tsimerman, 2014)

For $n \geq 1$ there exist only finitely many multi-modular n-tuples.

(Ineffective)

Related results

Theorem (Bilu–Masser–Zannier, 2013)

There are no solutions to $xy = 1$ in singular moduli.

Theorem (Bilu–Luca–Pizarro-Madariaga, arXiv 2014)

Explicit list of all solutions to $xy \in \mathbb{Q}^\times$.

Theorem (Habegger, arXiv 2014)

Only finitely many singular moduli are algebraic units.

Theorem (Bilu, Luca, Masser, arXiv 2015)

Only finitely many collinear triples of singular moduli.

AO for $\mathbb{C}^n$: for any particular equation $x_1^{a_1} \ldots x_n^{a_n} = 1$, only finitely many families of solutions.

Zilber-Pink/ "unlikely intersection" setting

Let $X = X_n = \mathbb{C}^n \times (\mathbb{C}^\times)^n$

Special subvarieties in $\mathbb{C}^n$: modular subvarieties M as above

Special subvarieties in $(\mathbb{C}^\times)^n$: "torsion cosets" $=$ cosets T of subtori by torsion points

Special subvarieties in X: Those of form $M \times T$.

Weakly special subvarieties in $\mathbb{C}^n$: allow arbitrary $x_i = \mathrm{constant}$ (not only $x_i = \sigma, \sigma \in \Sigma$), and (any) modular relations on M'

Weakly special subvarieties in $(\mathbb{C}^\times)^n$: cosets T' of subtori.

Weakly special subvarieties in X: Those of form $M' \times T'$.

Multi-modular tuples are "atypical"

Let $V = V_n \subset X$ given by

$$V = \{(x_1, \ldots, x_n; t_1, \ldots, t_n) : x_i = t_i, i = 1, \ldots, n\}.$$

Let $P = (\sigma_1, \ldots, \sigma_n) \in \Sigma^n$ be a multi-modular n-tuple.

So P lies in a proper special subvariety $T \subset (\mathbb{C}^\times)^n$.

Then $(P, P) \in V$ and it lies in a special subvariety $\{P\} \times T$ of X of codimension $n + 1$.

As $\dim V = n$, this is "atypical".

ZP

Sources: Zilber, Bombieri-Masser-Zannier, Pink.

For $V, W \subset X$, $A \subset V \cap W$ is **atypical in dimension** if

$$\dim A > \dim V + \dim W - \dim X.$$

Conjecture (ZP)

Let X be a variety of "mixed Shimura" type, and $V \subset X$. There is a **finite subset** *$\mathcal{S}_V$ of proper special subvarieties such that if S is a special subvariety and $A \subset V \cap S$ is atypical then $A \subset B$ for some atypical $B \subset V \cap T$ for some $T \in \mathcal{S}_V$.*

I.e. V has only finitely many maximal atypical subvarieties.

ZP implies AO, ML, and much more (and is very much open).

ZP and multi-modular n-tuples

ZP implies: only finitely many **isolated** multi-modular n-tuples (outside higher-dimensional atypical intersections). However:

Proposition

A multi-modular n-tuple cannot lie in a positive-dimensional atypical subvariety of V.

Proof. Suppose $P \in A \subset V \cap M \times T$, with A atypical and positive dimensional. By minimality of the n-tuple, T is codimension 1. Then $M \times T$ intersects V atypically iff $M \subset T$ when considering $M, T \subset (\mathbb{C}^{\times})^n$. So the conclusion follows from the following:

Multiplicative relations among $j(g_i z), g_i \in \mathrm{GL}_2^+(\mathbb{Q})$

Theorem

Let $g_1, \ldots, g_n \in \mathrm{GL}_2^+(\mathbb{Q})$. If the functions $j(g_i z)$ are distinct then they are **multiplicatively independent modulo constants**.

I.e. no $\prod j(g_i z)^{a_i} = c \in \mathbb{C}$, where $a_i \in \mathbb{Z}$ are not all zero.

Proof.

There is $z \in \mathbb{H}$ where $j(g_1 z) = 0$ but others non-zero. To see this, embed $\mathrm{PSL}_2(\mathbb{Z})\backslash\mathrm{PGL}_2^+(\mathbb{Q})$ in various $\mathrm{PSL}_2(\mathbb{Z}_p)\backslash\mathrm{PGL}_2(\mathbb{Q}_p)$ and use tree structure. □

"Complexity"

Suppose $P = (\sigma_1, \ldots, \sigma_n)$ is a multi-modular n-tuple.

Definition

The **complexity** of P is the maximum of the $|D(\tau_i)|$ where $j(\tau_i) = \sigma_i$ and $D(\tau_i)$ is the discriminant $b^2 - 4ac$ of the minimal (quadratic) polynomial of τ_i over $\mathbb{Z}$.

CM theory: $[\mathbb{Q}(\sigma_i) : \mathbb{Q}] = \#\mathrm{Cl}(D(\tau_i))$.

Siegel: $\#\mathrm{Cl}(D(\tau_i)) \gg_\delta |D|^{1/2-\delta}$ (ineffective)

So P has "many" conjugates.

Also: $\#\mathrm{Cl}(D(\tau_i)) \ll_\delta |D|^{1/2+\delta}$ (effective).

Also: Logarithmic Weil height $h(\sigma_i) \ll_\epsilon |D(\tau_i)|^\epsilon$.

The multiplicative relation...

...is controlled by the complexity:

Theorem (Yu, from Loher-Masser 2004)

Let $\alpha_1, \dots, \alpha_n \in K, [K : \mathbb{Q}] = d \geq 2$ *by multiplicatively dependent, but suppose no proper subset of them is. Then there is a non-trivial relation* $\prod \alpha_i^{b_i} = 1$ *with*

$$|b_i| \leq c(n) d^n \log d h(\alpha_1) \dots h(\alpha_n)/h(\alpha_i).$$

So: $|a_i| \leq c(n)\Delta(P)^{n(n+1)}$ (say) for the multi relation on P.

O-minimality and rational points

$$\pi : \mathbb{H}^n \times \mathbb{C}^n \to X$$

$$\pi(z_1, \ldots, z_n, u_1, \ldots, u_n) = (j(z_1), \ldots, \exp(u_1), \ldots)$$

Let $F_j, F_{\exp}$ be the standard fundamental domains. Then

$$Z = \pi^{-1}(V) \cap F_j^n \times F_{\exp}^n$$

is a **definable** set in the **o-minimal structure** $\mathbb{R}_{\mathrm{an\ exp}}$. So is:

$$Y = \{(z, u, t) \in Z \times \mathbb{R}^{n+1} : \sum u_i t_i = 2\pi i t_0\}$$

and its image Y' under projection to $\mathbb{H}^n \times \mathbb{R}^{n+1}$.

A multi-modular n-tuple P leads (via the point $(P, P) \in V$) to a "quadratic-rational" point in Y'.

Point-counting

A multi-modular P of complexity Δ has $\gg \Delta^{1/4}$ (say) conjugates, each gives a point in Y' which is quadratic in the $\mathbb{H}$ coords, rational in $\mathbb{R}^{n+1}$ coords, of (absolute) height at most $\ll \Delta^{n(n+1)}$.

The **Counting Theorem** (+ Alex Wilkie): A definable set in an o-minimal structure has $\ll_\epsilon T^\epsilon$ rational points up to (absolute) height T which don't lie on a connected positive-dimensional real algebraic subset.

Conclude: If $\Delta(P)$ is sufficiently large, there is a (real) algebraic curve in Y', giving a (non-constant) curve in $\mathbb{H}^n$ and associated hyperplanes which intersect Z.

The complex envelope of these gives: a **complex** algebraic set $W \subset \mathbb{H}^n \times \mathbb{C}^n$ of (complex) dimension n which intersects $\pi^{-1}(V)$ in a set of (complex) dimension 1.

This is "atypical" (in a different sense).

Ax-Schanuel

For (cartesian powers of) the **exponential** function:

Theorem (Ax, 1971)

Let $\Gamma \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ *be the graph of* exp, $V \subset \mathbb{C}^n \times (\mathbb{C}^\times)^n$ *an algebraic variety and* $A \subset \Gamma \cap V$ *an irreducible component. Then*

$$\dim A = \dim \Gamma + \dim V - 2n = \dim V - n$$

unless $\pi_{\mathbb{C}^n} A$ *is contained in a proper weakly special subvariety.*

I.e. considering the functions $z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}$ on A, have

$$\dim V = \mathrm{tr.deg.}_{\mathbb{C}} \mathbb{C}(z_1, \ldots, z_n, e^{z_1}, \ldots, e^{z_n}) \geq n + \dim A$$

unless $z_1, \ldots, z_n$ are "linearly dependent over $\mathbb{Q}$ mod $\mathbb{C}$" (i.e. some $\sum q_i z_i = c, q_i \in \mathbb{Q}, c \in \mathbb{C}$ holds on A). Note that there are $\dim A$ independent derivations on these functions.

“Two-sorted” Ax-Schanuel

AS implies: if $W \subset \mathbb{C}^n, V \subset (\mathbb{C}^\times)^n, A \subset W \cap \exp^{-1}(V)$ then

$$\dim A \leq \dim V + \dim W - n$$

unless A is contained in a proper weakly special subvariety.

One can eventually find a weakly special subvariety U' containing A such that the intersection is no longer atypical: with $X' = \exp U', V' = V \cap X'$ (we may assume $W \subset U'$)

$$\dim X' - \dim V' = \dim W' - \dim A.$$

I.e. the component $\pi^{-1}(V')$ has same “codimension” in U' as has A in W.

Modular Ax-Schanuel

For (cartesian powers of) the **modular function** $j : \mathbb{H} \to \mathbb{C}$:

Theorem (+Jacob Tsimerman, 2014)

Let $\Gamma \subset \mathbb{H}^n \times \mathbb{C}^n$ be the graph of j, $V \subset \mathbb{C}^n \times \mathbb{C}^n$ a subvariety, $A \subset \Gamma \cap V$ a component. Then

$$\dim A = \dim \Gamma + \dim V - 2n = \dim V - n$$

unless $\pi_{\mathbb{H}^n}(A)$ is contained in a proper weakly special subvariety.

(Even a version involving j', j''). Uses:

Complex geometry (Hwang-To)
O-minimal ("tame") complex geometry (Peterzil-Starchenko)
Monodromy (Deligne-André)
Point-counting in o-minimal structures

Conclusion

Deduce: "two-sorted" version for $(j, \exp) : \mathbb{H}^n \times \mathbb{C}^n \to \mathbb{C}^n \times (\mathbb{C}^\times)^n$.

Same implication: an "atypical" $W \cap j^{-1}(V)$ leads to a larger atypical intersection with a weakly special subvariety.

Proof of Theorem on multi-modular *n*-tuples. Sufficiently large $\Delta(P)$ for a multi-modular n-tuple P leads to "too many" points in Y' coming from conjugates of $(P, P) \in V$, and to an algebraic $W \subset \mathbb{H}^n \times \mathbb{C}^n$ which intersects $j^{-1}(V)$ "atypically".

Via "Ax-Schanuel" this leads to a positive dimensional weakly special subvariety which intersects atypically and contains some of the original multi-modular points and their associated special subvarieties. So we get a multi-modular tuple in a positive dimensional atypical intersection. Contradiction.

So $\Delta(P)$ is bounded, giving finiteness. □

Towards ZP for V_3

The remaining obstacle is (with variants):

Problem

Show that there are only finitely many triples (x, y, z) of distinct algebraic numbers which are pairwise multiplicatively dependent and pairwise "isogenous".

Problem: suitable bounds for heights of relation and degrees.

Conjecture

If (x, y) is such that $\Phi_N(x, y) = 0, x^a y^b = \zeta \in 1^{1/c}, (a, b) = 1$ then $h(x), h(y) \ll_\delta \max(N, |a|, |b|, c)^\delta$ for all $\delta > 0$.

This would suffice. Cf "Bounded Height Conjecture" of BMZ (theorem of Habegger) and conjectures of Habegger ("Weakly bounded height on modular curves"). Similar obstacle to ZP for curves in $\mathbb{C}^n$ (JP+Philipp Habegger).

Non-interaction of special structures

Looking beyond the "multiplicative independence mod constants" of $j(g_i z)$, one might conjecture (a la "Ax-Schanuel") that modular and multiplicative weakly specials "don't interact".

Conjecture

If $A \subset M \cap T$ where M is a weakly special modular subvariety, and T a weakly special multiplicative subvariety, both considered in $(\mathbb{C}^\times)^n$, then

$$\dim A \leq \dim M + \dim T - n$$

unless some coordinate is constant, or two coordinates equal on A.

Relations of the form $x_\ell = c$ and $x_i = x_j$ are the only ones common to both structures.

Modular ZP

+Philipp Habegger: full ZP for $Y(1)^n$ would follow from a suitable Galois orbit statement (and Modular Ax-Schanuel +Tsimerman).

Conjecture (Large Galois Orbit Conjecture; LGO)

Let $V \subset Y(1)^n$, defined over a f.g. field K. There are $c, \delta > 0$ such that if $P \in V$ is an optimal point then

$$[K(P) : K] \geq c\Delta(\langle P \rangle)^{\delta}$$

Here: $\langle A \rangle$ is the smallest special subvariety containing A, and $\Delta(A)$ is its complexity. E.g. $\Delta(\Phi_N(x, y) = 0) = N$.
The defect of A is $\delta(A) = \dim\langle A \rangle - \dim A$.
P is optimal if it is an irreducible component of $V \cap \langle P \rangle$, and no larger $A \subset V$ has same (or lower) defect.

Follows from a suitable height upper bound conjecture.

ZP for some curves

A curve $C \subset \mathbb{C}^n$ is called **asymmetric** if among the positive $\deg_{X_i} C$ there is at most one repetition.

Theorem (+Habegger, 2012)

Let $C \subset Y(1)^n$ be curve which is asymmetric and defined over $\overline{\mathbb{Q}}$. Then ZP holds for C.

Proof. Because we can prove LGO for asymmetric curves. □

Theorem

Let $C \subset Y(1)^3$ be a curve which is **not** *defined over $\overline{\mathbb{Q}}$. Then ZP holds for C.*

Proof. For **non-algebraic** x, y with $\Phi_N(x, y) = 0$, $[\mathbb{Q}(x, y) : \mathbb{Q}(x)]$ is "large". C is defined over some f.g. field K. Use: gonality of modular curve (Zograf-Abramovich), and point-counting. □