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Asymptotic Diophantine approximations

Let ξ be a real irrational and let $\psi : [1, \infty) \to (0, \infty)$ be a monotone increasing function. We consider the set

$$E_\xi(\psi, Q) = \{(p, q) \in \mathbf{Z}^2; |p + q\xi| < \frac{\psi(Q)}{Q}, 0 < q < Q\},$$

and we study its cardinality $|E_\xi(\psi, Q)|$ as Q becomes large. In this talk we present the following result: if $|p/q + \xi| \gg q^{-\beta}$ for all $(p, q) \in \mathbf{Z} \times \mathbf{N}$ then

$$|E_\xi(\psi, Q)| \sim 2\psi(Q) \quad \text{as } Q \to \infty$$

for every monotone increasing function $\psi : [1, \infty) \to (0, \infty)$ with $\psi(Q)/Q^{1-\frac{1}{\beta-1}} \to \infty$. The exponent $1 - \frac{1}{\beta-1}$ is sharp. A similar result holds in the inhomogeneous case and a slightly weaker one if the counting is restricted to coprime p, q. Both results generalise to simultaneous approximations. When time permits, we also explain how the same techniques can be applied to get asymptotics in the multiplicative setting, that is, for the cardinality of

$$\{q \in \mathbf{Z}; \|q\xi_1\|\|q\xi_2\| < \frac{\psi(Q)}{Q}, 0 < q < Q\}.$$

Here ξ_1 and ξ_2 are distinct badly approximable numbers, ψ is unbounded, and $\|\cdot\|$ is the distance to a nearest integer.